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Abstract: We discuss various physics aspects of neutrino oscillation with non-standard interactions (NSI). We formulate a perturbative framework by taking $\Delta m_{21}^2/\Delta m_{31}^2$, s_{13}, and the NSI elements $\varepsilon_{\alpha\beta}$ $(\alpha, \beta = e, \mu, \tau)$ as small expansion parameters of the same order ϵ. Within the ϵ perturbation theory we obtain the S matrix elements and the neutrino oscillation probability formula to second order (third order in ν_e related channels) in ϵ. The formula allows us to estimate size of the contribution of any particular NSI element $\varepsilon_{\alpha\beta}$ to the oscillation probability in arbitrary channels, and gives a global bird-eye view of the neutrino oscillation phenomena with NSI. Based on the second-order formula we discuss how all the conventional lepton mixing as well as NSI parameters can be determined. Our results shows that while θ_{13}, δ, and the NSI elements in ν_e sector can in principle be determined, complete measurement of the NSI parameters in the $\nu_\mu - \nu_\tau$ sector is not possible by the rate only analysis. The discussion for parameter determination and the analysis based on the matter perturbation theory indicate that the parameter degeneracy prevails with the NSI parameters. In addition, a new solar-atmospheric variable exchange degeneracy is found. Some general properties of neutrino oscillation with and without NSI are also illuminated.

Keywords: Neutrino Physics, Beyond Standard Model

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Contents

1 Introduction

Neutrino masses and lepton flavor mixing [1] discovered by the atmospheric [2], the so-lar [3], and the reactor neutrino [4] experiments constitute still the uniques evidence for physics beyond the Standard Model. A possible next step would be a discovery of neutrino interactions outside the standard electroweak theory. Based on expectation of new physics at TeV scale such non-standard interactions (NSI) with matter possessed by neutrinos are proposed and extensively discussed [5–10]. The experimental constraints on NSI are summarized in [11]. See also [12].

Recognition of structure of neutrino masses and lepton flavor mixing, at least up to now, relies on neutrino flavor transformation [1, 5, 13, 14], which we generically refer as neutrino oscillation in this paper. Quite naturally, there have been numerous theoretical analyses to understand the structure of the phenomena. In the context of long-baseline neutrino experiments, an exact expression of the oscillation probability is derived under the constant matter density approximation [15]. To understand physics of neutrino oscillation, however, it is often more illuminating to have suitable approximation schemes. In the latter category, various perturbative formulations of three-flavor neutrino oscillation have been developed and proven to be quite useful in particular in the context of long-baseline accel-erator and reactor experiments. They include one-mass scale dominance approximation in vacuum [16], short-distance expansion in matter [17], matter perturbation theory [18, 19], and perturbation theory with the small expansion parameters $\Delta m_{21}^2/\Delta m_{31}^2$ and θ_{13} [20] (that are taken as of order ϵ) which we call the ϵ perturbation theory in this paper. See, for example, [21–24] for subsequent development of perturbation theory of neutrino oscillation.

When the effects of NSI are included, however, theoretical analysis of the system of neutrino flavor transformation does not appears to achieve the same level of completeness

as that only with standard interactions (SI). Perturbative formulas of the oscillation probabilities with NSI have been derived under various assumptions [25–31]. Even some exact formulas are known [31]. However, one cannot answer the questions such as: How large is the effects of $\epsilon_{\mu\tau}$ in the oscillation probability $P(\nu_e \to \nu_\mu)$? (See below for definition of NSI elements $\epsilon_{\alpha\beta}$.) How large is the effects of ϵ_{ee} in the oscillation probability $P(\nu_e \to \nu_e)$? Which set of measurement is sufficient to determine all the NSI elements? References of neutrino oscillation and the sensitivity analyses with NSI are too numerous to quote here and may be found in bibliographies in the existing literatures, for example, in [30–32].

It is the purpose of this paper to fill the gap between understanding of neutrino oscillations with and without NSI. We try to do it by formulating the similar ϵ perturbation theory as in [20] but with including effects of NSI by assuming that NSI elements are of order $\sim \epsilon$. We derive the perturbative formula of the oscillation probability similar to the one in [20], which we call, respectively, the NSI and the SI second-order formulas in this paper. The approximate formula will allow us to have a bird-eye view of the neutrino oscillations with NSI, and will enable us to answer the above questions.

The other limitation that is present in some foregoing analyses, which we want to overcome, is the assumption of single (or, a few) $\epsilon_{\alpha\beta}$ dominance. Upon identification or getting hint for possible NSI interactions it will become possible to express $\varepsilon_{\alpha\beta}$ in propagation as a function of couplings involved in the higher dimensional operators. When this situation comes it is likely that all (or at least most of) the NSI elements $\varepsilon_{\alpha\beta}$ exist in the Hamiltonian with comparable magnitudes. Therefore, the theoretical machinery we prepare for the analysis must include all the NSI elements at the same time.[1]

More about necessity and usefulness of the S matrix and the NSI second-order formula of all oscillation channels and with all NSI elements included; If we are to include the effects of NSI in production and detection processes it is necessary to sum up all the oscillation channels that can contribute. Hence, the formulas of all channels are necessary. In a previous paper it was uncovered that the so called $\theta_{13}-$NSI confusion [26, 27] can be resolved by a two-detector setting in neutrino factory experiments [31]. Keeping the terms with the solar Δm_{21}^2 is shown to be crucial for resolving the confusion, and hence a full second-order formula is useful. In fact, the NSI second-order formula is surprisingly simple in its form, keeping the form of the original SI one with generalized variables, and the structure is even more transparent than those with first-order approximation of NSI.

With the NSI second order formula, we are able to discuss, for the first time, a strategy for simultaneous complete determination of the SI and NSI parameters. Through the course of discussions we indicate that, as in the system without NSI, the parameter degeneracy [33–35] exist in systems with NSI, but in a new form which involve both the SI and the NSI parameters. See sections 7 and 8. Moreover, we will uncover a new type of degeneracy, the one exchanging the generalized solar and atmospheric variables in section 7.7.

Finally, we should mention about what will not be achieved in this paper even within the context of theoretical analysis. First of all, our perturbative formulation relies on the particular assumption on relative magnitudes of SI and NSI parameters, and we cannot say

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[1]The similar comments also apply to the procedure by which the current constraints on NSI is derived (for example in [11]) where the constraints are derived under the assumption of presence of a particular NSI element in each time, the point carefully mentioned by the authors themselves.

many for cases in which our assumptions are not valid. We discuss the effects of NSI while neutrinos propagate in matter, and its effects in production and detection of neutrinos are ignored. Therefore, this paper must be regarded as merely the first step toward complete treatment of neutrino oscillation with NSI.

2 Physics summary

Because this paper has been developed into a long one, unfortunately, we think it convenient for readers, in particular experimentalists, to summarize the physics outputs of the perturbative treatment of neutrino oscillation with NSI. We highly recommend the readers to read this section first.

2.1 New result in the standard three flavor mixing

Though this paper aims at uncovering structure of neutrino oscillation with NSI, we have observed a new features of standard neutrino oscillation without NSI in section 5.3, the property we call the "matter hesitation". It states that in our perturbative framework the matter effect comes in into the oscillation probability only at the second order in the small expansion parameter ϵ in all the channels of neutrino oscillation.[2] It is a highly nontrivial feature because we treat the matter effect as of order unity. The "matter hesitation" explains why it is so difficult to have a sufficiently large matter effect, e.g., to resolve the mass hierarchy, in many long-baseline neutrino oscillation experiments. It also has implications to neutrino oscillation with NSI as will be discussed in section 5.4.

2.2 Guide for experimentalists; importance of various NSI elements in each channel

Experimentalists who want to hunt NSI in neutrino propagation may ask the following questions:

- We want to uncover the effect of $\varepsilon_{e\tau}$ (or $\varepsilon_{e\mu}$). What is the neutrino oscillation channel do you recommend to use for this purpose?

- We plan to detect the effect of $\varepsilon_{\mu\tau}$. Which set of measurement do we need to prepare?

- We seek a complete determination of all the SI and the NSI parameters. What would be the global strategy to adopt?

With the oscillation probability formulas given in section 6 we will try to answer these questions. Though we can offer only a partial answer to the last question above we can certainly give the answer to the first two questions within the framework of perturbation theory we use. In table 1 the relative importance of the effects of each element $\varepsilon_{\alpha\beta}$ of NSI are tabulated as order of a small parameter ϵ that they first appear in each oscillation channel. We presume $\epsilon \sim 10^{-2}$. Thus, our answer to the above questions based on the assumption that only the terms up to second order in ϵ are relevant would be (in order):

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[2]Though this property must be known in the community as the results of perturbative calculation, it appears to us that it did not receive enough attention so far.

- The neutrino oscillation channels in which only $\varepsilon_{e\tau}$ and $\varepsilon_{e\mu}$ come in and the other elements do not are the ν_e related ones, $\nu_e \to \nu_e$, $\nu_e \to \nu_\mu$, and $\nu_e \to \nu_\tau$. Obviously, the latter two appearance channels would be more interesting experimentally. One can in principle determine them simultaneously with θ_{13} and δ by rate only analysis.

- Do measurement at the $\nu_\mu \to \nu_\mu$ channel to determine $\varepsilon_{\mu\tau}$. Adding $\nu_\mu \to \nu_\tau$ channel does not help. Effects of the NSI element are relatively large because they are first order in ϵ, but the spectrum information is crucial to utilize this feature and to separate its effects from those of $\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}$. If an extreme precision is required you might want to supplement the measurement by the ν_μ and ν_τ appearance measurement above.

- We will show that, in fact, there is a difficulty in complete determination of all the NSI and SI parameters by the rate only analysis. The trouble occurs in the $\nu_\mu - \nu_\tau$ sector. Even though we are allowed to assume perfect measurement of all the channels including the one with ν_τ beam (which, of course, would not be practical), one of the three unknowns, $\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}$ and $\varepsilon_{\mu\tau}$ including its phase, cannot be determined if we rely on the rate only analysis. See section 7 for more details. Clearly, the spectrum information is the key to the potential of being able to determine all the SI and the NSI parameters, which should be taken into account in considering future facilities which search for NSI.

With regard to the second point above, some remarks are in order; Usually, disappearance channels are disadvantageous in looking for a small effect such as θ_{13}, because one has to make the statistical error smaller than the effect one wants to detect. In this respect, the NSI search in the $\nu_\mu - \nu_\tau$ sector is promising because it is the first order effect in ϵ. In fact, rather high sensitivities for determining $\varepsilon_{\mu\tau}$ and $\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}$ observed in atmospheric [36] and future accelerator [37] neutrino analyses are benefited by this feature.

We must warn the readers that experimental observable will be affected by NSI effects in production and detection of neutrinos. Therefore, our comments in this subsection assumes that they are well under control and shown to be smaller than the NSI effects in propagation by near detector measurement with an extreme precision. It should be also emphasized that some of our comments rely on the second-order perturbative formula of the oscillation probability.

2.3 Some interesting or peculiar features of neutrino oscillation with NSI

We list here some interesting features of neutrino oscillation with NSI which will be fully discussed in the following sections in this paper. Some of them are either unexpected, or might be showed up in previous analyses but without particular attention. A few points in them requires further investigation for full understanding.

- One of the most significant feature in table. 1 is that ε_{ee} appears only at third order in ϵ in all oscillation channels. It will be shown in section 5.4 that this feature can be explained as a consequence of the matter hesitation mentioned earlier.

- It is interesting to observe from table 1 that Wolfenstein's matter effect coefficient a in the oscillation probability, shows up in first (second) order in ϵ in system with

Channel	ϵ_{ee}	$\epsilon_{e\mu}$	$\epsilon_{e\tau}$	$\epsilon_{\mu\tau}$	$\epsilon_{\mu\mu}$	$\epsilon_{\tau\tau}$	a dep. (NSI)	a dep. (SI)
$P(\nu_e \to \nu_\alpha)$: $\alpha = e, \mu, \tau$	ϵ^3	ϵ^2	ϵ^2	ϵ^3	ϵ^3	ϵ^3	ϵ^2	ϵ^2
$P(\nu_\alpha \to \nu_\beta)$: $\alpha, \beta = \mu, \tau$	ϵ^3	ϵ^2	ϵ^2	ϵ^1	$\epsilon^1(\epsilon^2)$	$\epsilon^1(\epsilon^2)$	ϵ^1	ϵ^2

Table 1. Presented are the order in ϵ ($\sim 10^{-2}$) at which each type of $\varepsilon_{\alpha\beta}$ ($\alpha, \beta = e, \mu, \tau$) and a dependence (a is Wolfenstein's matter effect coefficient [5]) starts to come in into the expression of the oscillation probability in ϵ perturbation theory.[3] The last column is for the a dependence in the standard oscillation without NSI. The order of ϵ indicated in parentheses implies the one for the maximal θ_{23} in which cancellation takes place in the leading order. See the text for the definition of ϵ perturbation theory and for more details.

(without) NSI, which makes effects of matter density uncertainty larger in system with NSI. It occurs in the $\nu_\mu - \nu_\tau$ sector, and is easily understood as a consequence of "tree level" transition by the NSI element.

- The results in the last column in table 1 indicates that sensitivity to $\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}$ will depend upon if θ_{23} is maximal or not. This feature is clearly seen, e.g., in [37]. Analysis to resolve the θ_{23} octant degeneracy similar to the one proposed for cases without NSI [38–42], would be required for correct estimation of the sensitivity to NSI.

3 Introducing the effects of NSI in neutrino production, propagation and detection processes

We consider NSI involving neutrinos of the type

$$\mathcal{L}_{\text{eff}}^{\text{NSI}} = -2\sqrt{2}\, \varepsilon_{\alpha\beta}^{fP} G_F (\overline{\nu}_\alpha \gamma_\mu P_L \nu_\beta) (\overline{f} \gamma^\mu P f), \tag{3.1}$$

where G_F is the Fermi constant, and f stands for the index running over fermion species in the earth, $f = e, u, d$, where P stands for a projection operator and is either $P_L \equiv \frac{1}{2}(1 - \gamma_5)$ or $P_R \equiv \frac{1}{2}(1 + \gamma_5)$. The current constraints on ε parameters are summarized in [11].

Upon introduction of the NSI as in (3.1) it affects neutrino production, detection as well as propagation in matter [9, 25, 27, 28]. Therefore, we have to analyze the following "grand transition amplitude" from a parent Π particles (which needs not to be pions) to the particular detection particle N (which needs not be nucleons):[4]

$$T(E_\Pi, E_N) = \sum_{\alpha,\beta} \int dE_{\nu\alpha} D(E_\Pi, E_{\nu\alpha}) S(\nu_\alpha \to \nu_\beta; E_{\nu\alpha}) R(E_{\nu\beta}, E_N) \tag{3.2}$$

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[3]To second order in ϵ the sensitivity to $\varepsilon_{\mu\mu}$ and $\varepsilon_{\tau\tau}$ is through the form $\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}$, and hence no sensitivity to the individual ε's. Generally, the diagonal ε's appear in a form of difference in the oscillation probabilities because an over-all phase is an unobservable.

[4] One can talk about momentum reconstructed detected neutrinos instead of detected positrons, for example, but the reconstruction process must involve the effects of NSI. The expression in (3.2) is just to symbolically indicate this point.

where the sum over α and β must be taken only if they are amenable to be produced by the decay, or to undergo the reaction. Here, we have assumed the particular decay process to produce neutrinos as $\Pi \to \nu_\alpha + X_\alpha$ with decay amplitude $D(E_\Pi, E_{\nu\alpha})$ with the energies E_Π and $E_{\nu\alpha}$ of parent and daughter particles, and the particular reaction $\nu_\beta + P_{TG} \to N_\beta + Y_\beta$ with reaction amplitude $R(E_{\nu\beta}, E_N)$ which produce N_β particle with energy E_N. Here, X_α and Y_β are meant to be some inclusive collections of particles and P_{TG} denotes the target particle. $S(\nu_\alpha \to \nu_\beta)$ denotes the neutrino oscillation amplitude of the channel $\nu_\alpha \to \nu_\beta$. The observable quantity is of course $|T(E_\Pi, E_N)|^2$.

We assume that the coupling constant $\varepsilon_{\alpha\beta}$ possessed by NSI is small, $\sim \left(\frac{M_W}{M_{NP}}\right)^2$ where M_{NP} is a new physics scale, so that we can organize perturbative treatment of the effects of NSI. $\varepsilon_{\alpha\beta}$ can be as small as 10^{-2} (10^{-4}) for $M_{NP} = 1(10)\,\text{TeV}$, and is even smaller if higher dimension operators (higher than six) are required. We assume that all the $\varepsilon_{\alpha\beta}$ have similar order of magnitudes and denote the small number collectively as ϵ. Under these assumptions we expect that the decay and the detection functions, and the oscillation probabilities can be expanded as

$$D(E_\Pi, E_{\nu\alpha}) = D^{(0)} + D^{(1)}\epsilon + D^{(2)}\epsilon^2 + \cdots$$
$$S(\nu_\alpha \to \nu_\beta; E_{\nu\alpha}) = S^{(0)}(\nu_\alpha \to \nu_\beta) + S^{(1)}(\nu_\alpha \to \nu_\beta)\epsilon + S^{(2)}(\nu_\alpha \to \nu_\beta)\epsilon^2 + \cdots$$
$$R(E_{\nu\alpha}, E_N) = R^{(0)} + R^{(1)}\epsilon + R^{(2)}\epsilon^2 + \cdots \tag{3.3}$$

where we have suppressed the kinematical dependences in quantities in the right-hand-side of (3.3). The first terms in (3.3) are the one without NSI. Now, because of the smallness of $\epsilon \sim 10^{-2}$ (or smaller) we take the attitude that keeping terms up to second order in ϵ must be good enough to discuss the effects of NSI and eventually to estimate the sensitivity to NSI.[5]

Unfortunately, even with the perturbative treatment this is a highly complicated system to analyze its full structure. It is possible that the types of NSI that contribute to production and detection processes are more numerous than the ones in the propagation process [30]. If this occurs the effects of NSI into production and detection processes could be qualitatively different from those in propagation. Therefore, the effects of NSI come into the decay and the reaction amplitudes generally in a model-dependent fashion, so that the flavor (α, β) dependence of NSI effects are also model-dependent. Also they do so in an energy dependent way so that integration over neutrino energy in (3.2) is required for the full analysis. For an explicit example of how NSI enter into the decay and the reaction amplitudes as well as to the neutrino propagation in matter in concrete models, see for example the "unitarity violation" approach developed in [43].

In this paper, therefore, we confine ourselves to analysis of the structure of neutrino propagation with NSI, namely the terms with no effects of NSI in the decay and the reaction amplitudes in (3.2). This is a particularly simple system (relatively speaking with the full one) in the sense that no unitarity violation comes in because it deals with propagation of three light neutrinos. Furthermore, it has no explicit model dependence once the effects

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[5]As far as the appearance channels $\nu_e \to \nu_\mu$ and $\nu_e \to \nu_\tau$ are concerned the oscillation amplitudes start from first order in ϵ, as we will see below. Therefore, only the first order corrections to D and R are relevant for the observable to order ϵ^2.

of NSI is parametrized in the familiar way. See the Hamiltonian in (4.2). We should emphasize that limitation of our scope to the problem of neutrino propagation, in fact, allows us to dig out structure of neutrino oscillation with NSI in a transparent manner. Therefore, we think it a meaningful first step.

Our analysis can become the whole story provided that extremely stringent bounds on NSI effects in decay as well as detection reactions are placed by front detector measurement in future experiments. Otherwise, it covers only a leading (zeroth) order terms in NSI effect in decay and detection. When the first order corrections to them are taken into account what is needed is to compute the oscillation amplitude up to first order in ϵ to obtain the observable to order ϵ^2. Hence, we present the results of S matrix elements in appendix A, not only the expression of the oscillation probabilities, for future use.

4 General properties of neutrino oscillation with and without NSI

Now, we analyze the structure and the properties of neutrino propagation in matter with NSI. We, however, sometimes go back to the system without NSI whenever it is illuminating. The results obtained in this section are exact, that is, they are valid without recourse to perturbation theory we will formulate in the next section. To discuss effects of NSI on neutrino propagation it is customary to introduce the ε parameters, which are defined as $\varepsilon_{\alpha\beta} \equiv \sum_{f,P} \frac{n_f}{n_e} \varepsilon_{\alpha\beta}^{fP}$, where n_f (n_e) denotes the f-type fermion (electron) number density along the neutrino trajectory in the earth. Then, the neutrino evolution equation can be written in flavor basis as

$$i\frac{d}{dx}\nu_\alpha = H_{\alpha\beta}\nu_\beta \qquad (\alpha, \beta = e, \mu, \tau). \tag{4.1}$$

In the standard three-flavor neutrino scheme, Hamiltonian including NSI is given by

$$H = \frac{1}{2E} \left\{ U \begin{bmatrix} 0 & 0 & 0 \\ 0 & \Delta m_{21}^2 & 0 \\ 0 & 0 & \Delta m_{31}^2 \end{bmatrix} U^\dagger + a(x) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right.$$
$$\left. + a(x) \begin{bmatrix} \varepsilon_{ee} & |\varepsilon_{e\mu}|e^{i\phi_{e\mu}} & |\varepsilon_{e\tau}|e^{i\phi_{e\tau}} \\ |\varepsilon_{e\mu}|e^{-i\phi_{e\mu}} & \varepsilon_{\mu\mu} & |\varepsilon_{\mu\tau}|e^{i\phi_{\mu\tau}} \\ |\varepsilon_{e\tau}|e^{-i\phi_{e\tau}} & |\varepsilon_{\mu\tau}|e^{-i\phi_{\mu\tau}} & \varepsilon_{\tau\tau} \end{bmatrix} \right\} \tag{4.2}$$

where $\Delta m_{ji}^2 \equiv m_j^2 - m_i^2$, and $a(x) \equiv 2\sqrt{2}G_F N_e(x)E$ is the coefficient which is related to the index of refraction of neutrinos in medium of electron number density $N_e(x)$ [5], where G_F is the Fermi constant and E is the neutrino energy. The first two terms in (4.2) are the Standard Model interactions, whereas the last term denotes the non-standard neutrino interactions with matter. U denotes the flavor mixing matrix, the Maki-Nakagawa-Sakata (MNS) matrix [1], in the lepton sector. In its standard form [44] it reads

$$U = U_{23}U_{13}U_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{bmatrix} \begin{bmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{bmatrix} \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.3}$$

where δ stands for the leptonic Kobayashi-Maskawa (KM) phase, and c_{ij} and s_{ij} denote $\cos\theta_{ij}$ and $\sin\theta_{ij}$, respectively.

Most of the formulas in this and the next sections (sections 4 and 5) can be written in forms valid for arbitrary matter density profiles if the adiabatic approximation holds. We, however, present the ones derived under the constant matter density approximation because it makes the equations simpler, in particular, the perturbative formulas for the oscillation probabilities in section 6. Unlike the case of the MSW solar neutrino solutions [14] in which the matter density variation is the key to the problem, the constant density approximation to $N_e(x)$ in long-baseline experiments should serve as a reasonable first approximation.

The S matrix describes possible flavor changes after traversing a distance L,

$$\nu_\alpha(L) = S_{\alpha\beta}\nu_\beta(0), \tag{4.4}$$

and the oscillation probability is given by

$$P(\nu_\beta \to \nu_\alpha; L) = |S_{\alpha\beta}|^2. \tag{4.5}$$

If the neutrino evolution is governed by the Schrödinger equation (4.1), S matrix is given as

$$S = T\exp\left[-i\int_0^L dx H(x)\right] \tag{4.6}$$

where T symbol indicates the "time ordering" (in fact "space ordering" here). The right-hand-side of (4.6) may be written as e^{-iHL} for the case of constant matter density. For notational convenience, we denote the S matrix elements as

$$S = \begin{bmatrix} S_{ee} & S_{e\mu} & S_{e\tau} \\ S_{\mu e} & S_{\mu\mu} & S_{\mu\tau} \\ S_{\tau e} & S_{\tau\mu} & S_{\tau\tau} \end{bmatrix}. \tag{4.7}$$

The primary purpose of this paper is to discuss the properties of neutrino oscillation in the standard three flavor system with NSI. But, we recollect the properties of neutrino oscillation without NSI whenever necessary, and treat both systems simultaneously or go back and forth between them to make our discussion transparent. By this way the properties of the neutrino oscillations can be better illuminated.

4.1 Relations between neutrino oscillation amplitudes without NSI

If NSI, the third term in (4.2), is absent the matter term (the second term in (4.2)) has a symmetry; It is invariant under U_{23} rotation which act on $\nu_\mu - \nu_\tau$ subspace. Due to this symmetry the Hamiltonian can be conveniently written in the form

$$H = U_{23}\tilde{H}U_{23}^\dagger, \tag{4.8}$$

and hence the S matrix can be written as

$$S(L) = U_{23}\tilde{S}(L)U_{23}^\dagger \tag{4.9}$$

as noticed in [23] where $\tilde{S}(L) = T\exp\left[-i\int_0^L dx\,\tilde{H}(x)\right]$. The point here is that $\tilde{H}$ and $\tilde{S}(L)$ do *not* contain θ_{23}.

If we denote $\tilde{S}(L)$ matrix elements in a form analogous to the one in (4.7) S matrix can be written as

$$
\begin{bmatrix}
\tilde{S}_{ee} & c_{23}\tilde{S}_{e\mu}+s_{23}\tilde{S}_{e\tau} & -s_{23}\tilde{S}_{e\mu}+c_{23}\tilde{S}_{e\tau} \\
c_{23}\tilde{S}_{\mu e}+s_{23}\tilde{S}_{\tau e} & c_{23}^2\tilde{S}_{\mu\mu}+s_{23}^2\tilde{S}_{\tau\tau}+c_{23}s_{23}(\tilde{S}_{\mu\tau}+\tilde{S}_{\tau\mu}) & c_{23}^2\tilde{S}_{\mu\tau}-s_{23}^2\tilde{S}_{\tau\mu}+c_{23}s_{23}(\tilde{S}_{\tau\tau}-\tilde{S}_{\mu\mu}) \\
-s_{23}\tilde{S}_{\mu e}+c_{23}\tilde{S}_{\tau e} & c_{23}^2\tilde{S}_{\tau\mu}-s_{23}^2\tilde{S}_{\mu\tau}+c_{23}s_{23}(\tilde{S}_{\tau\tau}-\tilde{S}_{\mu\mu}) & s_{23}^2\tilde{S}_{\mu\mu}+c_{23}^2\tilde{S}_{\tau\tau}-c_{23}s_{23}(\tilde{S}_{\mu\tau}+\tilde{S}_{\tau\mu})
\end{bmatrix}.
$$

(4.10)

It should be noticed that $S_{ee} = \tilde{S}_{ee}$ is independent of θ_{23}. Therefore, the S matrix elements obey relationships [23]

$$
\begin{aligned}
S_{e\tau} &= S_{e\mu}(c_{23}\to -s_{23}, s_{23}\to c_{23}), \\
S_{\tau\tau} &= S_{\mu\mu}(c_{23}\to -s_{23}, s_{23}\to c_{23}), \\
S_{\tau\mu} &= -S_{\mu\tau}(c_{23}\to -s_{23}, s_{23}\to c_{23}).
\end{aligned}
$$

(4.11)

4.2 Relations between neutrino oscillation amplitudes with NSI

The secret behind the relations between $S_{e\mu}$ and $S_{e\tau}$ and the others in (4.11) is that $\tilde{H}$ is independent of θ_{23}, or in other words, the invariance of $\tilde{H}$ under the transformation $c_{23}\to -s_{23}, s_{23}\to c_{23}$. When NSI is introduced there exists the following additional term in $\tilde{H}$:

$$
\begin{aligned}
\tilde{H}^{\rm NSI} &= U_{23}^\dagger
\begin{bmatrix}
\varepsilon_{ee} & \varepsilon_{e\mu} & \varepsilon_{e\tau} \\
\varepsilon_{e\mu}^* & \varepsilon_{\mu\mu} & \varepsilon_{\mu\tau} \\
\varepsilon_{e\tau}^* & \varepsilon_{\mu\tau}^* & \varepsilon_{\tau\tau}
\end{bmatrix} U_{23}
\equiv
\begin{bmatrix}
\tilde{\varepsilon}_{ee} & \tilde{\varepsilon}_{e\mu} & \tilde{\varepsilon}_{e\tau} \\
\tilde{\varepsilon}_{e\mu}^* & \tilde{\varepsilon}_{\mu\mu} & \tilde{\varepsilon}_{\mu\tau} \\
\tilde{\varepsilon}_{e\tau}^* & \tilde{\varepsilon}_{\mu\tau}^* & \tilde{\varepsilon}_{\tau\tau}
\end{bmatrix} \\
&=
\begin{bmatrix}
\varepsilon_{ee} & c_{23}\varepsilon_{e\mu}-s_{23}\varepsilon_{e\tau} & s_{23}\varepsilon_{e\mu}+c_{23}\varepsilon_{e\tau} \\
c_{23}\varepsilon_{e\mu}^*-s_{23}\varepsilon_{e\tau}^* & c_{23}^2\varepsilon_{\mu\mu}+s_{23}^2\varepsilon_{\tau\tau}-c_{23}s_{23}(\varepsilon_{\mu\tau}+\varepsilon_{\mu\tau}^*) & c_{23}^2\varepsilon_{\mu\tau}-s_{23}^2\varepsilon_{\mu\tau}^*+c_{23}s_{23}(\varepsilon_{\mu\mu}-\varepsilon_{\tau\tau}) \\
s_{23}\varepsilon_{e\mu}^*+c_{23}\varepsilon_{e\tau}^* & c_{23}^2\varepsilon_{\mu\tau}^*-s_{23}^2\varepsilon_{\mu\tau}+c_{23}s_{23}(\varepsilon_{\mu\mu}-\varepsilon_{\tau\tau}) & s_{23}^2\varepsilon_{\mu\mu}+c_{23}^2\varepsilon_{\tau\tau}+c_{23}s_{23}(\varepsilon_{\mu\tau}+\varepsilon_{\mu\tau}^*)
\end{bmatrix}
\end{aligned}
$$

(4.12)

Because $\tilde{H}^{\rm NSI}$ in (4.12) *does* depend on θ_{23}, the S matrix relations as given in (4.11) do not hold. However, if we consider the extended transformation

$$
\begin{aligned}
c_{23} &\to -s_{23}, & s_{23} &\to c_{23}, \\
\varepsilon_{e\mu} &\to \varepsilon_{e\tau}, & \varepsilon_{e\tau} &\to -\varepsilon_{e\mu}, \\
\varepsilon_{\mu\mu} &\to \varepsilon_{\tau\tau}, & \varepsilon_{\tau\tau} &\to \varepsilon_{\mu\mu}, \\
\varepsilon_{\mu\tau} &\to -\varepsilon_{\mu\tau}^*, & \varepsilon_{\mu\tau}^* &\to -\varepsilon_{\mu\tau},
\end{aligned}
$$

(4.13)

it is easy to show that $\tilde{H}^{\rm NSI}$ is invariant under the transformation (4.13). It means that the S matrix relations (4.11) hold even with NSI provided that we extend the transformation to the ones in (4.13). It not only implies the existence of useful relations between the S matrix elements, but also serves as a powerful tool for consistency check of perturbative computation. We will see in appendices. A and B that the computed results of the S matrix elements, and hence the oscillation probabilities, do satisfy (4.11) with the extended transformation (4.13).

As we will see in section 6.2, the invariance under the extended transformation (4.13) entails a remarkable feature that the terms which depend on ε's in the $\nu_\mu - \nu_\tau$ sector in the oscillation probabilities $P(\nu_\mu \to \nu_\mu)$, $P(\nu_\mu \to \nu_\tau)$, and $P(\nu_\tau \to \nu_\tau)$ are all equal up to sign.

4.3 Phase reduction theorem

Now, we present a general theorem on reduction of number of CP violating phases in system with NSI, which we call "phase reduction theorem" for short. By looking into the results of perturbative computation [28] it was observed that when the solar Δm_{21}^2 is switched off the oscillation probabilities with NSI depends on phases which come from NSI elements and δ in a particular manner, e.g., $|\varepsilon| e^{i(\delta + \phi)}$. It was conjectured on physics ground that the property must hold in the exact expressions of the oscillation probabilities [31]; With vanishing Δm_{21}^2 the system becomes effectively two flavor and hence the observable CP violating phase must be unique.

Here, we give a general proof of this property which is, in fact, very easy to do. We first notice a simple relation which holds in the absence of Δm_{21}^2,

$$
\hat{H} \equiv \begin{bmatrix} e^{i\delta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} H \begin{bmatrix} e^{-i\delta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.14}
$$

$$
= \Delta \begin{bmatrix} s_{13}^2 & c_{13}s_{13}s_{23} & c_{13}s_{13}c_{23} \\ c_{13}s_{13}s_{23} & c_{13}^2 s_{23}^2 & c_{13}^2 c_{23}s_{23} \\ c_{13}s_{13}c_{23} & c_{13}^2 c_{23}s_{23} & c_{13}^2 c_{23}^2 \end{bmatrix} + \frac{a}{2E} \begin{bmatrix} 1 + \varepsilon_{ee} & |\varepsilon_{e\mu}|e^{i\chi} & |\varepsilon_{e\tau}|e^{i\omega} \\ |\varepsilon_{e\mu}|e^{-i\chi} & \varepsilon_{\mu\mu} & |\varepsilon_{\mu\tau}|e^{i\phi_{\mu\tau}} \\ |\varepsilon_{e\tau}|e^{-i\omega} & |\varepsilon_{\mu\tau}|e^{-i\phi_{\mu\tau}} & \varepsilon_{\tau\tau} \end{bmatrix},
$$

where $\Delta \equiv \frac{\Delta m_{31}^2}{2E}$, $\chi \equiv \delta + \phi_{e\mu}$, and $\omega \equiv \delta + \phi_{e\tau}$. Then, if we use a new basis $\hat{\nu}_\alpha \equiv [\text{diag}(e^{i\delta}, 1, 1)]_{\alpha\beta} \nu_\beta$, the evolution equation reads

$$
i\frac{d}{dx} \begin{bmatrix} \hat{\nu}_e \\ \hat{\nu}_\mu \\ \hat{\nu}_\tau \end{bmatrix} = \hat{H} \begin{bmatrix} \hat{\nu}_e \\ \hat{\nu}_\mu \\ \hat{\nu}_\tau \end{bmatrix}. \tag{4.15}
$$

It is obvious from (4.15) that the system depends on only three phases $\chi = \delta + \phi_{e\mu}$, $\omega = \delta + \phi_{e\tau}$ and $\phi_{\mu\tau}$ out of four. This particular combination of phases is, of course, depends upon the specific parametrization of the MNS matrix. The phase factor attached to the transformation matrix in (4.14) does not affect the oscillation probability because it is an over-all phase.

The similar treatment with the same transformation as in (4.14) (diag.$(1, 1, e^{-i\delta})$) can be used to prove that the phase reduction occurs if $\theta_{12} = 0$ ($\theta_{23} = 0$) even though $\Delta m_{21}^2 \neq 0$.[6] If $\theta_{13} = 0$ it is obvious that there are no effect of δ.

This completes a general proof that number of CP violating phases is reduced by one when the solar Δm_{21}^2 is switched off, or one of the mixing angles vanishes. We emphasize that this property has implications to the real world; For example, the phenomenon of phase reduction occurs at the magic baseline, $\frac{aL}{4E} = \pi$, in the perturbative formula to be obtained in section 6 even though $\Delta m_{21}^2 \neq 0$.

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[6]We thank Hiroshi Nunokawa for calling our attention to this feature.

5 Perturbation theory of neutrino oscillation

5.1 ϵ perturbation theory

To formulate perturbation theory one has to specify the expansion parameters. We take the following dimensionless parameters as small expansion parameters and assume that they are of the same order:[7]

$$\frac{\Delta m_{21}^2}{\Delta m_{31}^2} \sim s_{13} \sim \varepsilon_{\alpha\beta} \sim \epsilon \quad (\alpha, \beta = e, \mu, \tau). \tag{5.1}$$

Whereas, we treat $\frac{a}{\Delta m_{31}^2}$ and $\frac{\Delta m_{31}^2 L}{2E}$ as of order unity. We collectively denote order of magnitude of the expansion parameters as ϵ, and hence we call the perturbative framework the ϵ perturbation theory. In the absence of NSI our formulas of oscillation probabilities, of course, reduces to the Cervera et $al.$ formula [20], which we call the SI second-order formula in this paper. Correspondingly, we call our second-order probability formula the "NSI second-order formula". It appears that in the standard case this perturbative framework accommodates the situation of relatively large θ_{13} within the Chooz bound [46], and applicable to wide variety of experimental settings.

Another approach would be to just expand in terms of $\varepsilon_{\alpha\beta}$ which is assumed to be small without any correlation with other SI mixing parameters. If NSI elements are extremely small, much smaller than the SI expansion parameters, such first-order formulas of NSI would be sufficient. It would be the case of NSI search in the next generation experiments as discussed e.g., in [30].

On the contrary, it often occurs in deriving constraints on various NSI parameters that the bounds on the diagonal ε's, ε_{ee}, $\varepsilon_{\mu\mu}$, and $\varepsilon_{\tau\tau}$, are sometimes milder than the ones on the off-diagonal ε's by an order of magnitude. If it is the case, we may need to keep the higher order of the diagonal ε's in ϵ-perturbation theory, to e.g. ϵ^4, in probabilities to analyze such situations. We try not to enter into this problem in our present treatment.

5.2 Formulating perturbative framework

We follow the standard perturbative formulation to calculate the S matrix and the neutrino oscillation probabilities [18]. Yet, we present a simplified treatment which is suitable for higher order calculations. For convenience, we start by treating the system without NSI in this section. We use the tilde-basis $\tilde{\nu} = U_{23}^\dagger \nu$ with Hamiltonian $\tilde{H}$ defined in (4.8). The tilde-Hamiltonian is decomposed as $\tilde{H} = \tilde{H}_0 + \tilde{H}_1$, where

$$\tilde{H}_0(x) = \Delta \begin{bmatrix} r_A(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.2}$$

$$\tilde{H}_1 = \Delta \begin{bmatrix} s_{13}^2 & 0 & c_{13}s_{13}e^{-i\delta} \\ 0 & 0 & 0 \\ c_{13}s_{13}e^{i\delta} & 0 & -s_{13}^2 \end{bmatrix} + \Delta r_\Delta \begin{bmatrix} s_{12}^2 c_{13}^2 & c_{12}s_{12}c_{13} & -s_{12}^2 c_{13}s_{13}e^{-i\delta} \\ c_{12}s_{12}c_{13} & c_{12}^2 & -c_{12}s_{12}s_{13}e^{-i\delta} \\ -s_{12}^2 c_{13}s_{13}e^{i\delta} & -c_{12}s_{12}s_{13}e^{i\delta} & s_{12}^2 s_{13}^2 \end{bmatrix} \tag{5.3}$$

---

[7]We do not take $\frac{1}{\sqrt{2}} - s_{23}$ as an expansion parameter because a rather large range is currently allowed and the situation will not be changed even with the next generation experiments [24].

where $\Delta \equiv \frac{\Delta m_{31}^2}{2E}$ $r_\Delta \equiv \frac{\Delta m_{21}^2}{\Delta m_{31}^2}$, $r_A(x) \equiv \frac{a(x)}{\Delta m_{31}^2}$. Though our treatment can be easily generalized to cases with matter density variation as far as the adiabatic approximation holds, we present, for ease of presentation, the formulas with constant matter density approximation.

To calculate $\tilde{S}(L)$ we define $\Omega(x)$ as

$$\Omega(x) = e^{i\tilde{H}_0 x}\tilde{S}(x). \tag{5.4}$$

$\Omega(x)$ obeys the evolution equation

$$i\frac{d}{dx}\Omega(x) = H_1\Omega(x) \tag{5.5}$$

where

$$H_1 \equiv e^{i\tilde{H}_0 x}\tilde{H}_1 e^{-i\tilde{H}_0 x} \tag{5.6}$$

Then, $\Omega(x)$ can be computed perturbatively as

$$\Omega(x) = 1 + (-i)\int_0^x dx' H_1(x') + (-i)^2 \int_0^x dx' H_1(x') \int_0^{x'} dx'' H_1(x'') + \mathcal{O}(\epsilon^3). \tag{5.7}$$

where the "space-ordered" form in (5.7) is essential because of the highly nontrivial spatial dependence in H_1. Collecting the formulas the S matrix can be written as

$$S(L) = U_{23}e^{-i\tilde{H}_0 L}\Omega(L)U_{23}^\dagger \tag{5.8}$$

Therefore, essentially we are left with perturbative computation of $\Omega(x)$ with use of (5.6) to calculate the S matrix.[8]

5.3 Matter hesitation and unitarity

One of the usefulness of the ϵ perturbation theory is that it allows to prove the property "matter hesitation", which is a characteristic feature of neutrino oscillation in matter without NSI in small θ_{13} regime. The matter hesitation refers to the property that the matter effect dependent terms in the neutrino oscillation probabilities $P(\nu_\alpha \to \nu_\beta)$ $(\alpha, \beta = e, \mu, \tau)$ are absent to first order in ϵ. Namely, it hesitates to come in before computation goes to second order in ϵ. Though its validity heavily relies on the particular perturbative framework we work in this paper, it explains why it is so difficult to detect the matter effect in many accelerator experiments.

In fact, it is easy to observe the property of matter hesitation; It directly follows from the structure of the S matrix in (5.8) itself. We first note that in the tilde-basis $\tilde{H}_1$ is free from the matter effect and it exists only in $\tilde{H}_0$. Therefore, the matter effect dependence exists only in $e\mu$ and $e\tau$ (and their conjugate) elements in H_1 in (5.6), and they are of order ϵ. The same statement follows for Ω in (5.7). Then, the matter effect dependence in $\tilde{S} \equiv e^{-i\tilde{H}_0 L}\Omega$ is only in $e\mu$, $e\tau$ and ee elements at first order in ϵ. Notice that the final

---

[8]Since $\tilde{H}_1$ in (5.3) contains order ϵ^2 terms in addition to order ϵ terms the formal expression in (5.7) includes terms higher than $\mathcal{O}(\epsilon^2)$ which are meant to be ignored. The same statement applies to the computation to be carried out in section 6.

rotation in 23 space to obtain the S matrix in (5.8) does not alter the property of the $\tilde{S}$ matrix. Therefore, no matter effect dependence appears in the oscillation probabilities to first order in ϵ in ee, $e\mu - e\tau$ and in the $\mu\tau$ sector channels for different reasons: In $P(\nu_\mu \to \nu_\mu)$ and $P(\nu_\mu \to \nu_\tau)$ the matter effect is trivially absent to order ϵ because of no dependence in the S matrix. In $P(\nu_e \to \nu_\mu)$ and $P(\nu_e \to \nu_\tau)$ it comes in only at order ϵ^2 because the S matrix elements are of order ϵ. In $P(\nu_e \to \nu_e)$ the matter effect is absent to order ϵ because it is contained in a phase factor of the S matrix element. This completes the derivation of the matter hesitation, the property that matter effects comes in into the oscillation probability only at second order in ϵ.

We stress that the absence of the matter effect in the oscillation probability to first order in ϵ is highly nontrivial, in particular in S_{ee}. Since the matter effect coefficient $r_A = \frac{a}{\Delta m_{31}^2}$ is zeroth order in ϵ it can affect the S matrix in all orders of ϵ. But, in fact, absence of matter effect to first order in ϵ in $P(\nu_e \to \nu_e)$ can be understood by unitarity. For the most nontrivial channel, the relevant unitarity relation is

$$1 - P(\nu_e \to \nu_e) = P(\nu_e \to \nu_\mu) + P(\nu_e \to \nu_\tau) \tag{5.9}$$

Since $P(\nu_e \to \nu_\mu)$ and $P(\nu_e \to \nu_\tau)$ are at least of order ϵ^2 as will be shown in section 6 and appendix B, the matter dependent term in $P(\nu_e \to \nu_e)$, which is involved in the left-hand-side in (5.9), has to be second order, or higher, in ϵ. It should be noticed that this argument is valid not only in systems with SI only but also in the one with NSI, the matter hesitation property for $P(\nu_e \to \nu_e)$ in the presence of NSI. Also notice that the same argument does not go through for $1 - P(\nu_\mu \to \nu_\mu)$ because $P(\nu_\mu \to \nu_\tau)$ can contain the terms lower than ϵ^2. If fact, there exists the first order term in ϵ in $P(\nu_\mu \to \nu_\mu)$ and $P(\nu_\mu \to \nu_\tau)$ which are proportional to the matter effect coefficient a.

5.4 Implication of matter hesitation to neutrino oscillation with NSI

There is a clear implication of the property of matter hesitation to the system with NSI; The terms with the NSI element ε_{ee} must appear in the oscillation probability only at third-order in ϵ or higher. It is due to the special nature of ε_{ee} that can be introduced as a renormalization factor of the matter effect coefficient a, $a \to a(1 + \varepsilon_{ee})$. Since the terms with a are already of order ϵ^2, the terms with ε_{ee} must be at least of order ϵ^3.

The reader should be puzzled by the above statement. One may argue quite naturally that there must exist a term with first order in ε_{ee} in the survival probability $P(\nu_e \to \nu_e)$. In fact, such a term does exist in the relevant S matrix element as one can see in (A.1):

$$S_{ee} = e^{-ir_A\Delta L}\Big\{ 1 - i(s_{12}^2 r_\Delta + \varepsilon_{ee} r_A)\Delta L \Big\} \tag{5.10}$$

Resolution of the puzzle, therefore, is that the first order term of ε_{ee} cannot appears in the oscillation probability because it is purely imaginary, or a phase ignoring ϵ^2 terms. However, to confirm the cancellation of second order term we must go beyond the present treatment by keeping the order ϵ^2 terms in the S matrix. It will be done in the next section.

A more general question is whether the matter hesitation can be generalized into the whole systems with NSI. We have already answered the question at the end of the previous subsection. This feature is to be verified by explicit computation in section 6.

6 NSI second-order probability formulas

Now, we present the expressions of the oscillation probabilities with NSI which is valid to second order in ϵ. For ease of computation we use a slightly different basis which we call the double-tilde basis with Hamiltonian

$$H = U_{23}U_{13}\tilde{\tilde{H}}U_{13}^\dagger U_{23}^\dagger \tag{6.1}$$

and the corresponding S matrix

$$S(L) = U_{23}U_{13}\tilde{\tilde{S}}(L)U_{13}^\dagger U_{23}^\dagger \tag{6.2}$$

where $\tilde{\tilde{S}}(L) = T\exp\left[-i\int_0^L dx\tilde{\tilde{H}}(x)\right]$. The zeroth order and the perturbed part of the reduced Hamiltonian $\tilde{\tilde{H}}$ are given by

$$\tilde{\tilde{H}}_0 = \Delta \begin{bmatrix} r_A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.3}$$

$$\tilde{\tilde{H}}_1 = \Delta \left\{ r_\Delta \begin{bmatrix} s_{12}^2 & c_{12}s_{12} & 0 \\ c_{12}s_{12} & c_{12}^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + r_A \begin{bmatrix} -s_{13}^2 & 0 & c_{13}s_{13}e^{-i\delta} \\ 0 & 0 & 0 \\ c_{13}s_{13}e^{i\delta} & 0 & s_{13}^2 \end{bmatrix} \right\}$$
$$+ \Delta r_A U_{13}^\dagger \begin{bmatrix} \tilde{\varepsilon}_{ee} & \tilde{\varepsilon}_{e\mu} & \tilde{\varepsilon}_{e\tau} \\ \tilde{\varepsilon}_{e\mu}^* & \tilde{\varepsilon}_{\mu\mu} & \tilde{\varepsilon}_{\mu\tau} \\ \tilde{\varepsilon}_{e\tau}^* & \tilde{\varepsilon}_{\mu\tau}^* & \tilde{\varepsilon}_{\tau\tau} \end{bmatrix} U_{13} \tag{6.4}$$

where $\Delta \equiv \frac{\Delta m_{31}^2}{2E}$, $r_\Delta \equiv \frac{\Delta m_{21}^2}{\Delta m_{31}^2}$, $r_A \equiv \frac{a}{\Delta m_{31}^2}$. To simplify the expressions of the S matrix elements we use the NSI elements in the tilde basis, $\tilde{\varepsilon}_{\alpha\beta} = (U_{23}^\dagger)_{\alpha\gamma}\varepsilon_{\gamma\delta}(U_{23})_{\delta\beta}$, defined in (4.12). Notice that $\tilde{\varepsilon}$'s are invariant under the extended transformation (4.13).

The perturbative computation of the S matrix elements can be done with the formulas similar to the ones in the tilde basis in section 5.2. In this section we concentrate on the structural analysis of the NSI second-order oscillation probabilities e.g., for analysis of parameter determination. We collect all the resultant explicit expressions of S matrix elements and the oscillation probabilities in appendix A and B, respectively. The results of third-order calculation which are necessary to complete table 1 are presented in appendix C.

6.1 Electron neutrino sector

The most distinctive feature of the NSI second-order oscillation probabilities in the ν_e-related sector is that they have very similar forms as the SI second-order formulas [20] but with the generalized atmospheric and the solar variables:

$$\Theta_\pm \equiv s_{13}\frac{\Delta m_{31}^2}{a} + (s_{23}\varepsilon_{e\mu} + c_{23}\varepsilon_{e\tau})e^{i\delta} = \pm s_{13}\frac{\delta m_{31}^2}{a} + |\tilde{\varepsilon}_{e\tau}|e^{i\hat{\phi}_{e\tau}}$$
$$\Xi \equiv \left(c_{12}s_{12}\frac{\Delta m_{21}^2}{a} + c_{23}\varepsilon_{e\mu} - s_{23}\varepsilon_{e\tau}\right)e^{i\delta} = c_{12}s_{12}\frac{\Delta m_{21}^2}{a}e^{i\delta} + |\tilde{\varepsilon}_{e\mu}|e^{i\hat{\phi}_{e\mu}} \tag{6.5}$$

and their antineutrino versions

$$\bar{\Theta}_{\pm} \equiv -s_{13}\frac{\Delta m_{31}^2}{a} + (s_{23}\varepsilon_{e\mu}^* + c_{23}\varepsilon_{e\tau}^*)e^{-i\delta} = \mp s_{13}\frac{\delta m_{31}^2}{a} + |\tilde{\varepsilon}_{e\tau}|e^{-i\hat{\phi}_{e\tau}},$$

$$\bar{\Xi} \equiv \left(-c_{12}s_{12}\frac{\Delta m_{21}^2}{a} + c_{23}\varepsilon_{e\mu}^* - s_{23}\varepsilon_{e\tau}^*\right)e^{-i\delta} = -c_{12}s_{12}\frac{\Delta m_{21}^2}{a}e^{-i\delta} + |\tilde{\varepsilon}_{e\mu}|e^{-i\hat{\phi}_{e\mu}}, \quad (6.6)$$

where $\hat{\phi}_{e\alpha} \equiv \delta + \tilde{\phi}_{e\alpha}$ ($\alpha = \mu, \tau$). The particular dependence on NSI elements in (6.5) and (6.6) has root in the form of the perturbed Hamiltonian (4.12) in the tilde-basis, from which it can be understood that $\tilde{\varepsilon}_{e\mu}$ and $\tilde{\varepsilon}_{e\tau}$ play the role of the mixing angles which govern 1-2 and 1-3 transitions, respectively. At the second equality in the right-hand-side of these equations we have introduced a new notation $\Delta m_{31}^2 = \pm\delta m_{31}^2$ where $\pm$ sign indicates the sign of Δm_{31}^2, the mass hierarchy, and $\delta m_{31}^2 \equiv |\Delta m_{31}^2| > 0$. Note that $a \equiv 2\sqrt{2}G_F N_e E > 0$. For convenient notation we parametrize these quantities as

$$\Theta_{\pm} = |\Theta_{\pm}|e^{i\theta_{\pm}}, \qquad \Xi = |\Xi|e^{i\xi}.$$
$$\bar{\Theta}_{\pm} = |\bar{\Theta}_{\pm}|e^{i\bar{\theta}_{\pm}}, \qquad \bar{\Xi} = |\bar{\Xi}|e^{i\bar{\xi}}. \qquad (6.7)$$

To represent the oscillation probability in a compact way we define

$$X_{\pm} \equiv \left(\frac{a}{\delta m_{31}^2 \mp a}\right)^2 \sin^2\frac{\delta m_{31}^2 \mp a}{4E}L,$$

$$Y_{\pm} \equiv \left(\frac{a}{\delta m_{31}^2 \mp a}\right)\sin\frac{aL}{4E}\sin\frac{\delta m_{31}^2 \mp a}{4E}L,$$

$$Z \equiv \sin^2\frac{aL}{4E}. \qquad (6.8)$$

For anti-neutrinos we have flipped sign of a, and hence

$$\bar{X}_{\pm} = X_{\mp},$$
$$\bar{Y}_{\pm} = Y_{\mp}. \qquad (6.9)$$

Z is obviously invariant under the sign change of a, $\bar{Z} = Z$.

With these notations and by defining $\Delta_{31} \equiv \frac{\Delta m_{31}^2 L}{4E}$ for simplicity of expressions, the oscillation probabilities $P(\nu_e \to \nu_e)$, $P(\nu_e \to \nu_\mu)$ and $P(\nu_e \to \nu_\tau)$ (together with the anti-neutrino counterparts of the latter two) can be written as

$$P(\nu_e \to \nu_e) = 1 - 4X_{\pm}|\Theta_{\pm}|^2 - 4Z|\Xi|^2 \qquad (6.10)$$

$$P(\nu_e \to \nu_\mu) = 4s_{23}^2 X_{\pm}|\Theta_{\pm}|^2 + 4c_{23}^2 Z|\Xi|^2 + 8c_{23}s_{23}Y_{\pm}|\Xi||\Theta_{\pm}|\cos(\xi - \theta_{\pm} - |\Delta_{31}|) \quad (6.11)$$

$$P(\nu_e \to \nu_\tau) = 4c_{23}^2 X_{\pm}|\Theta_{\pm}|^2 + 4s_{23}^2 Z|\Xi|^2 - 8c_{23}s_{23}Y_{\pm}|\Xi||\Theta_{\pm}|\cos(\xi - \theta_{\pm} - |\Delta_{31}|) \quad (6.12)$$

$$P(\nu_\mu \to \nu_e) = T[P(\nu_e \to \nu_\mu)]$$
$$= 4s_{23}^2 X_{\pm}|\Theta_{\pm}|^2 + 4c_{23}^2 Z|\Xi|^2 + 8c_{23}s_{23}Y_{\pm}|\Xi||\Theta_{\pm}|\cos(\xi - \theta_{\pm} + |\Delta_{31}|) \quad (6.13)$$

$$P(\bar{\nu}_e \to \bar{\nu}_\mu) = \text{CP}[P(\nu_e \to \nu_\mu)]$$
$$= 4s_{23}^2 X_{\mp}|\bar{\Theta}_{\pm}|^2 + 4c_{23}^2 Z|\bar{\Xi}|^2 + 8c_{23}s_{23}Y_{\mp}|\bar{\Xi}||\bar{\Theta}_{\pm}|\cos(\bar{\xi} - \bar{\theta}_{\pm} - |\Delta_{31}|) \quad (6.14)$$

$$P(\bar{\nu}_e \to \bar{\nu}_\tau) = \text{CP}[P(\nu_e \to \nu_\tau)]$$
$$= 4c_{23}^2 X_{\mp}|\bar{\Theta}_{\pm}|^2 + 4s_{23}^2 Z|\bar{\Xi}|^2 - 8c_{23}s_{23}Y_{\mp}|\bar{\Xi}||\bar{\Theta}_{\pm}|\cos(\bar{\xi} - \bar{\theta}_{\pm} - |\Delta_{31}|) \quad (6.15)$$

$$P(\bar{\nu}_\mu \to \bar{\nu}_e) = \text{T}[P(\bar{\nu}_e \to \bar{\nu}_\mu)] = \text{TCP}[P(\nu_e \to \nu_\mu)]$$
$$= 4s_{23}^2 X_{\mp}|\bar{\Theta}_{\pm}|^2 + 4c_{23}^2 Z|\bar{\Xi}|^2 + 8c_{23}s_{23}Y_{\mp}|\bar{\Xi}||\bar{\Theta}_{\pm}|\cos(\bar{\xi} - \bar{\theta}_{\pm} + |\Delta_{31}|) \quad (6.16)$$

The upper and the lower signs in the above equations are for the normal and the inverted hierarchies, respectively. The expression of $P(\nu_e \to \nu_e)$ is so simple because of the unitarity, $P(\nu_e \to \nu_e) = 1 - [P(\nu_e \to \nu_\mu) + P(\nu_e \to \nu_\tau)]$.

Notice that to second order in ϵ, the oscillation probabilities in the ν_e related sector do not contain any NSI elements in the $\nu_\mu - \nu_\tau$ sector, $\varepsilon_{\mu\tau}$ etc. It should not come as a surprise because in the ν_μ and ν_τ appearance channel from ν_e the leading term of the S matrix is already of order ϵ, and it can contain only the ν_e related NSI elements, $\varepsilon_{e\mu}$ and $\varepsilon_{e\tau}$. Therefore, to order ϵ^2 there is no room for NSI elements in the $\nu_\mu - \nu_\tau$ sector in the appearance probabilities. We will see in the next subsection that this simple fact leads to a great simplification of the oscillation probabilities in the $\nu_\mu - \nu_\tau$ sector.

We note, in passing, that because of the relation $Y_\pm = \sqrt{X_\pm Z}$ which is easily recognized by (6.8) it is evident that the oscillation probabilities can be written in a form of absolute square of addition of the atmospheric and the solar terms.[9] For example, $P(\nu_e \to \nu_\mu)$ takes the form

$$P(\nu_e \to \nu_\mu) = 4\left| s_{23}\sqrt{X_\pm}|\Theta_\pm| + c_{23}\sqrt{Z}|\Xi|\exp\left[i\left(\xi - \theta_\pm - |\Delta_{31}|\right)\right] \right|^2. \tag{6.17}$$

At the magic baseline, $\frac{aL}{4\pi} = \pi$, the second term vanishes because $Z = 0$, leaving a very simple expression of the oscillation probability, $P(\nu_e \to \nu_\mu) = 4s_{23}^2 X_\pm |\Theta_\pm|^2$.

6.2 $\nu_\mu - \nu_\tau$ sector

As in the ν_e-related sector there is a distinct characteristic feature of the oscillation probabilities with NSI in the $\nu_\mu - \nu_\tau$ sector. Namely, to second order in ϵ they can be decomposed into the three pieces with different dependences on NSI elements, the vacuum term, the ones with $\varepsilon_{\alpha\beta}$ in the ν_e-related and the $\nu_\mu - \nu_\tau$ sectors, respectively:

$$\begin{aligned}
P(\nu_\alpha \to \nu_\beta; \varepsilon_{e\mu}, \varepsilon_{e\tau}, \varepsilon_{\mu\mu}, \varepsilon_{\mu\tau}, \varepsilon_{\tau\tau}) &= P(\nu_\alpha \to \nu_\beta; 2 \text{ flavor in vacuum}) \\
&\quad + P(\nu_\alpha \to \nu_\beta; \varepsilon_{e\mu}, \varepsilon_{e\tau}) \\
&\quad + P(\nu_\alpha \to \nu_\beta; \varepsilon_{\mu\mu}, \varepsilon_{\mu\tau}, \varepsilon_{\tau\tau})
\end{aligned} \tag{6.18}$$

where α and β denote one of μ and τ. The explicit expressions of these terms will be displayed in appendix B.

The point is that the last term in (6.18) is universal, up to sign, among all the three channels, $P(\nu_\mu \to \nu_\mu)$, $P(\nu_\mu \to \nu_\tau)$, and $P(\nu_\tau \to \nu_\tau)$. Though it may look mysterious, it is in fact very simple to understand it. By unitarity it follows that

$$\begin{aligned}
P(\nu_\mu \to \nu_\mu) + P(\nu_\mu \to \nu_\tau) &= 1 - P(\nu_\mu \to \nu_e), \\
P(\nu_\tau \to \nu_\tau) + P(\nu_\tau \to \nu_\mu) &= 1 - P(\nu_\tau \to \nu_e).
\end{aligned} \tag{6.19}$$

We note that $P(\nu_\mu \to \nu_e)$ and $P(\nu_\tau \to \nu_e)$ do not contain $\varepsilon_{\mu\mu}$, $\varepsilon_{\tau\tau}$, and $\varepsilon_{\mu\tau}$ to second order in ϵ. Then, it follows from the first equation in (6.19) that $P(\nu_\mu \to \nu_\tau; \varepsilon_{\mu\mu}, \varepsilon_{\mu\tau}, \varepsilon_{\tau\tau}) = -P(\nu_\mu \to \nu_\mu; \varepsilon_{\mu\mu}, \varepsilon_{\mu\tau}, \varepsilon_{\tau\tau})$. Noticing that the terms related to ε's in the $\nu_\mu - \nu_\tau$ sector are

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[9]We thank Stephen Parke for calling our attention to this point.

T-invariant, the relations $P(\nu_\tau \to \nu_\tau; \varepsilon_{\mu\mu}, \varepsilon_{\mu\tau}, \varepsilon_{\tau\tau}) = -P(\nu_\mu \to \nu_\tau; \varepsilon_{\mu\mu}, \varepsilon_{\mu\tau}, \varepsilon_{\tau\tau})$ must also hold. Therefore, the $\varepsilon_{\alpha\beta}$ $(\alpha, \beta = \mu, \tau)$ dependent term in the three channels are all equal up to sign. The equality $P(\nu_\mu \to \nu_\mu; \varepsilon_{\mu\mu}, \varepsilon_{\mu\tau}, \varepsilon_{\tau\tau}) = P(\nu_\tau \to \nu_\tau; \varepsilon_{\mu\mu}, \varepsilon_{\mu\tau}, \varepsilon_{\tau\tau})$ also follows from the relationship between S matrix elements due to the extended transformation (4.13).

7 Parameter determination in neutrino oscillation with NSI

Thanks to the NSI second-order probability formulas derived in the previous section, we can now address the question of how simultaneous measurement of the SI and the NSI parameters can be carried out. However, we must first warn the readers that our discussions in this section are based solely on the NSI second-order formulas, and hence its validity may be limited. Nonetheless, we believe that ignoring the ϵ^3 effects is quite safe because we anticipate $\epsilon \sim 10^{-2}$ in our perturbative framework.

7.1 SI-NSI confusion

One of the most distinctive features of the oscillation probability formulas in section 6 is that the NSI parameters $\varepsilon_{e\alpha}$ $(\alpha = \mu, \tau)$ appears in the particular combination with the SI parameters as in (6.5) and (6.6). What that means in the context of parameter determination? It means that, in general, determination of SI mixing parameters, θ_{13} and δ, has severe confusion with determination of NSI parameters $\varepsilon_{\alpha\beta}$, and vice versa. However, it should be noticed that it does *not* mean something like "No Go" theorem. Namely, there is a way to circumvent this problem. It is a complete determination of the SI and the NSI parameters, the possibility we address later in this section.

Nonetheless, we should note the following: If such complete determination is somehow not feasible experimentally, our result may be interpreted as an analytic proof of the "NSI-SI confusion theorem".[10] It is a powerful statement because it not only reveals the existence of confusion but also illuminates which SI parameters are confused with which NSI parameters via which manner.

In fact, the characteristic feature in (6.5), namely, $\tilde{\varepsilon}_{e\mu}$ only couples to the solar scale oscillation and $\tilde{\varepsilon}_{e\tau}$ the atmospheric one, would affect the resolution of the θ_{13}-NSI and the two-phase confusions. Coupling between the solar and the atmospheric degrees of freedom bridged by a NSI element is the key to the resolution of the θ_{13}-NSI confusion by the two-detector method [31]. Therefore, the resolution mechanism might be affected by the simultaneous presence of two ε's, which "decouples" the solar and the atmospheric degrees of freedom. This point deserves a careful investigation.

7.2 Strategy for parameter determination

To gain a hint of how we can proceed let us look at table. 1. We first note that it is not possible to detect the effects of ε_{ee} because it is of third order in all channels, and hence we have to omit it from our subsequent discussions.[11] It is also well known and is obvious

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[10]We note that a different type of the confusion theorem was derived in [27] which involves θ_{13} and NSI parameters in production and in propagation processes that obey a special relationship.

[11]If we take the setting with only ε_{ee} as NSI, it can be regarded as uncertainty in the matter density and it is known that neutrino factory has a great sensitivity to it [47, 48]. However, in our current setting the issue

from the probability formulas in appendix B that $\varepsilon_{\mu\mu}$ and $\varepsilon_{\tau\tau}$ come in through the form $\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}$ and therefore only their difference is measurable.

Next, we observe that in ν_μ and ν_τ appearance channels from ν_e, only the ν_e related NSI, $\varepsilon_{e\mu}$ and $\varepsilon_{e\tau}$ appear to second order in ϵ. Therefore, the obvious strategy is to use these channels for complete determination of them simultaneously with θ_{13} and δ. Then, we may be able to determine the rest of the NSI parameters in the $\nu_\mu - \nu_\tau$ sector by disappearance and appearance measurement in that sector.

The important point is therefore that one can explore the effects of $\varepsilon_{e\mu}$ and $\varepsilon_{e\tau}$ in ν_e related channels while ignoring $\varepsilon_{\mu\tau}$, $\varepsilon_{\mu\mu}$, and $\varepsilon_{\tau\tau}$. It is a good news because the appearance channels, assuming excellent detection capability of ν_μ and ν_τ, have great potential of detecting the effects of NSI [31]. Once $\varepsilon_{e\mu}$ and $\varepsilon_{e\tau}$ are measured one can proceed to determine the rest of the NSI elements $\varepsilon_{\mu\tau}$, $\varepsilon_{\mu\mu}$, and $\varepsilon_{\tau\tau}$ using the oscillation probabilities in the $\nu_\mu - \nu_\tau$ sector.[12]

7.3 Complete measurement of the SI and the NSI parameters; θ_{13}, δ, $\varepsilon_{e\mu}$ and $\varepsilon_{e\tau}$

Now, we start to formulate a recipe for complete determination of the SI and the NSI parameters. Based on consideration in the previous subsection, we concentrate on $P(\nu_e \to \nu_\mu)$ and $P(\nu_e \to \nu_\tau)$ and their CP and T conjugates. By looking into the expressions of oscillation probabilities in (6.11) and (6.12) (and other related ones which will be given below) one notices that the observable quantities are of the forms

$$|\Theta_\pm|^2, |\Xi|^2, \xi - \theta_\pm, \text{ in neutrino sector, and}$$
$$|\bar{\Theta}_\pm|^2, |\bar{\Xi}|^2, \bar{\xi} - \bar{\theta}_\pm, \text{ in antineutrino sector.} \tag{7.1}$$

where the phase ξ, $\theta_\pm$, etc are defined in (6.7). There are altogether six quantities.

Suppose now that somehow we were able to determine all these quantities. We discuss in the following subsections how it can be done. Here, we show how they determine the SI and the NSI parameters, s_{13}, δ, $|\tilde{\varepsilon}_{e\mu}|$, $|\tilde{\varepsilon}_{e\tau}|$, $\phi_{e\mu}$, and $\phi_{e\tau}$. It may be sufficient, assuming that the inversion is possible, to express the observable in terms of the physical parameters. We start with the neutrino sector:

$$|\Theta_\pm|^2 = s_{13}^2 \left(\frac{\delta m_{31}^2}{a}\right)^2 + |\tilde{\varepsilon}_{e\tau}|^2 \pm 2s_{13}|\tilde{\varepsilon}_{e\tau}| \left(\frac{\delta m_{31}^2}{a}\right) \cos\hat{\phi}_{e\tau},$$
$$|\Xi|^2 = \left(c_{12}s_{12}\frac{\Delta m_{21}^2}{a}\right)^2 + |\tilde{\varepsilon}_{e\mu}|^2 + 2c_{12}s_{12}|\tilde{\varepsilon}_{e\mu}|\frac{\Delta m_{21}^2}{a}\cos(\delta - \hat{\phi}_{e\mu}). \tag{7.2}$$

---

of matter density uncertainty is much more severe and universal; It produces uncertainties in determining all the NSI elements. Clearly, the discussion of this point is beyond the scope of the present paper.

[12]If θ_{23} is deviated significantly from the maximal so that $\cos 2\theta_{23} \gg \epsilon$, then the terms with $\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}$ can have sizes of order ϵ. In this case, it may be possible to detect the effects of $\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}$ and measure (or constrain) it even without having a priori knowledges of $\varepsilon_{e\mu}$ and $\varepsilon_{e\tau}$.

For phase difference we obtain

$$
\begin{aligned}
\frac{\Theta_\pm^* \Xi}{\Theta_\pm \Xi^*} &= e^{2i(\xi-\theta_\pm)} = \frac{1}{|\Theta_\pm|^2|\Xi|^2} \\
&\times \left[s_{13}^2 \left(\frac{\delta m_{31}^2}{a} \right)^2 \left\{ \left(c_{12}s_{12}\frac{\Delta m_{21}^2}{a} \right)^2 e^{2i\delta} + |\tilde{\varepsilon}_{e\mu}|^2 e^{2i\hat{\phi}_{e\mu}} + 2c_{12}s_{12}|\tilde{\varepsilon}_{e\mu}|\frac{\Delta m_{21}^2}{a}e^{i(\delta+\hat{\phi}_{e\mu})} \right\} \right. \\
&\quad + |\tilde{\varepsilon}_{e\tau}|^2 \left\{ \left(c_{12}s_{12}\frac{\Delta m_{21}^2}{a} \right)^2 e^{2i(\delta-\hat{\phi}_{e\tau})} + |\tilde{\varepsilon}_{e\mu}|^2 e^{2i(\hat{\phi}_{e\mu}-\hat{\phi}_{e\tau})} + 2c_{12}s_{12}|\tilde{\varepsilon}_{e\mu}|\frac{\Delta m_{21}^2}{a}e^{i(\delta+\hat{\phi}_{e\mu}-2\hat{\phi}_{e\tau})} \right\} \\
&\quad \pm 2s_{13}|\tilde{\varepsilon}_{e\tau}| \left(\frac{\delta m_{31}^2}{a} \right) \left\{ \left(c_{12}s_{12}\frac{\Delta m_{21}^2}{a} \right)^2 e^{i(2\delta-\hat{\phi}_{e\tau})} \right. \\
&\quad\quad \left.\left. + |\tilde{\varepsilon}_{e\mu}|^2 e^{i(2\hat{\phi}_{e\mu}-\hat{\phi}_{e\tau})} + 2c_{12}s_{12}|\tilde{\varepsilon}_{e\mu}|\frac{\Delta m_{21}^2}{a}e^{i(\delta+\hat{\phi}_{e\mu}-\hat{\phi}_{e\tau})} \right\} \right]. \quad (7.3)
\end{aligned}
$$

By taking the real and the imaginary parts of (7.3) one can obtain $\cos 2(\xi - \theta_\pm)$ and $\sin 2(\xi - \theta_\pm)$, respectively. For antineutrinos we obtain

$$
\begin{aligned}
|\bar{\Theta}_\pm|^2 &= s_{13}^2 \left(\frac{\delta m_{31}^2}{a} \right)^2 + |\tilde{\varepsilon}_{e\tau}|^2 \mp 2s_{13}|\tilde{\varepsilon}_{e\tau}| \left(\frac{\delta m_{31}^2}{a} \right) \cos\hat{\phi}_{e\tau}, \\
|\bar{\Xi}|^2 &= \left(c_{12}s_{12}\frac{\Delta m_{21}^2}{a} \right)^2 + |\tilde{\varepsilon}_{e\mu}|^2 - 2c_{12}s_{12}|\tilde{\varepsilon}_{e\mu}|\frac{\Delta m_{21}^2}{a}\cos(\delta - \hat{\phi}_{e\mu}). \quad (7.4)
\end{aligned}
$$

Similarly, the equation for the phase difference $\bar{\theta}_\pm - \bar{\xi}$ similar to (7.3) can be obtained by making the transformation $a \to -a$, $\delta \to -\delta$, $\hat{\phi}_{e\mu} \to -\hat{\phi}_{e\mu}$, $\hat{\phi}_{e\tau} \to -\hat{\phi}_{e\tau}$ in (7.3).

Having the six equations altogether with given six observable, $|\Theta_\pm|$, $|\bar{\Theta}_\pm|$, $|\Xi|$, $|\bar{\Xi}|$, $\bar{\xi} - \bar{\theta}_\pm$, and $\xi - \theta_\pm$, they can be solved for the six unknowns, s_{13}, δ, two complex numbers $\tilde{\varepsilon}_{e\mu}$, and $\tilde{\varepsilon}_{e\tau}$. Given the latter two numbers one can determine the original $\varepsilon_{e\mu}$ and $\varepsilon_{e\tau}$. Therefore, the rest of the problem in simultaneous determination of the SI and the NSI parameters is how to measure the above six observable.

7.4 Measurement with a monochromatic neutrino beam; ν_e sector

In this subsection, we discuss a way of determining the SI-NSI combined parameters in (7.1) by assuming a set of measurement at an energy E, aiming at their complete determination.[13] Though it might not be a practical way, by describing a concrete method we try to illuminate characteristic features of the problem of complete determination. With the six unknowns we have to prepare neutrino oscillation measurement of six different channels.

Suppose that one measures the following six probabilities at a neutrino energy E, $P(\nu_e \to \nu_\mu)$, $P(\nu_e \to \nu_\tau)$, $P(\nu_\mu \to \nu_e) = \mathrm{T}[P(\nu_e \to \nu_\mu)]$, $P(\bar{\nu}_e \to \bar{\nu}_\mu) = \mathrm{CP}[P(\nu_e \to \nu_\mu)]$, $P(\bar{\nu}_e \to \bar{\nu}_\tau) = \mathrm{CP}[P(\nu_e \to \nu_\tau)]$, $P(\bar{\nu}_\mu \to \bar{\nu}_e) = \mathrm{T}[P(\bar{\nu}_e \to \bar{\nu}_\mu)]$. Notice that we have intensionally avoided to use the channels which require ν_τ beam which, if not impossible,

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[13] It was proposed that such a monochromatic neutrino beam can be prepared for ν_e and $\bar{\nu}_e$ beams [49, 50].

would be very difficult to prepare. From (6.11), (6.12), and (6.13), it is easy to obtain

$$P(\nu_e \to \nu_\mu) + P(\nu_\mu \to \nu_e) = 8s_{23}^2 X_\pm |\Theta_\pm|^2 + 8c_{23}^2 Z|\Xi|^2$$
$$+16c_{23}s_{23}Y_\pm |\Xi||\Theta_\pm| \cos(\xi - \theta_\pm) \cos|\Delta_{31}| \quad (7.5)$$

$$P(\nu_e \to \nu_\mu) - P(\nu_\mu \to \nu_e) = 16c_{23}s_{23}Y_\pm |\Xi||\Theta_\pm| \sin(\xi - \theta_\pm) \sin|\Delta_{31}| \quad (7.6)$$

$$P(\nu_e \to \nu_\mu) + P(\nu_e \to \nu_\tau) = 4X_\pm |\Theta_\pm|^2 + 4Z|\Xi|^2 \quad (7.7)$$

Similarly, for the antineutrino channels, we obtain from (6.14), (6.15), and (6.16),

$$P(\bar{\nu}_e \to \bar{\nu}_\mu) + P(\bar{\nu}_\mu \to \bar{\nu}_e) = 8s_{23}^2 X_\mp |\bar{\Theta}_\pm|^2 + 8c_{23}^2 Z|\bar{\Xi}|^2$$
$$+16c_{23}s_{23}Y_\mp |\bar{\Xi}||\bar{\Theta}_\pm| \cos(\bar{\xi} - \bar{\theta}_\pm) \cos|\Delta_{31}| \quad (7.8)$$

$$P(\bar{\nu}_e \to \bar{\nu}_\mu) - P(\bar{\nu}_\mu \to \bar{\nu}_e) = 16c_{23}s_{23}Y_\mp |\bar{\Xi}||\bar{\Theta}_\pm| \sin(\bar{\xi} - \bar{\theta}_\pm) \sin|\Delta_{31}| \quad (7.9)$$

$$P(\bar{\nu}_e \to \bar{\nu}_\mu) + P(\bar{\nu}_e \to \bar{\nu}_\tau) = 4X_\mp |\bar{\Theta}_\pm|^2 + 4Z|\bar{\Xi}|^2 \quad (7.10)$$

It is easy to solve these equations to obtain $|\Theta_\pm|$, $|\Xi|$, and $(\xi - \theta_\pm)$ (for neutrinos), and $|\bar{\Theta}_\pm|$, $|\bar{\Xi}|$, and $(\bar{\xi} - \bar{\theta}_\pm)$ (for antineutrinos).

It may be obvious that the above analysis can be converted to the rate only analysis by replacing the probabilities $P(\nu_\alpha \to \nu_\beta)$ by energy integrated number of events with fluxes and cross sections $\int dE F_\alpha \sigma_{\nu N} P(\nu_\alpha \to \nu_\beta)$, and the similar integrated quantities of $X_\pm$ etc.

7.5 Determining the NSI parameters in the $\nu_\mu - \nu_\tau$ sector

After measurement of θ_{13}, δ, and $\varepsilon_{e\mu}$ and $\varepsilon_{e\tau}$ as described in the previous subsection, one can proceed to determination of the NSI parameters in the $\nu_\mu - \nu_\tau$ sector with (for concreteness) mono-energetic beam. As we saw in section 6.2 the $\varepsilon_{\mu\mu}$ and $\varepsilon_{\mu\tau}$ dependent term in the oscillation probabilities is universal in $P(\nu_\mu \to \nu_\mu)$, $P(\nu_\mu \to \nu_\tau)$, and $P(\nu_\tau \to \nu_\tau)$. Therefore, one can simply use one of the above three channels, which means that τ neutrino beam, even if it were prepared, does not help.

The oscillation probability $P(\nu_\mu \to \nu_\mu)$ derived in section 6 can be written as

$$P(\nu_\mu \to \nu_\mu; \varepsilon_{e\mu}, \varepsilon_{e\tau}, \varepsilon_{\mu\mu}, \varepsilon_{\mu\tau}, \varepsilon_{\tau\tau}) = P(\nu_\mu \to \nu_\mu; \varepsilon_{e\mu}, \varepsilon_{e\tau}) \quad (7.11)$$
$$+\mathcal{D}_\pm^{(0)}(\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}) + \mathcal{R}_\pm^{(0)}\mathrm{Re}(\varepsilon_{\mu\tau}) + \mathcal{D}^{(1)}(\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}) + \mathcal{R}^{(1)}\mathrm{Re}(\varepsilon_{\mu\tau})$$
$$+\mathcal{S}^{(0)}(\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau})^2 + \mathcal{W}^{(0)}(\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau})\mathrm{Re}(\varepsilon_{\mu\tau}) + \mathcal{Q}^{(0)}\mathrm{Re}(\varepsilon_{\mu\tau})^2 + \mathcal{I}^{(0)}\mathrm{Im}(\varepsilon_{\mu\tau})^2$$

where the explicit form of the coefficients can be easily read off from the expressions in appendix B and we have the similar expression for antineutrinos. We have obtained two equations for the three unknowns, $\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}$, $\mathrm{Re}(\varepsilon_{\mu\tau})$, and $\mathrm{Im}(\varepsilon_{\mu\tau})$. Clearly we need one more equation to determine the three unknowns, which is unavailable under the current setting. Thus, we have to conclude that a complete determination of the NSI elements in the $\nu_\mu - \nu_\tau$ sector is not possible by measurement at a monochromatic beam or the rate only analysis.

7.6 Necessity of spectrum analysis

Doing measurement at six different channels is *not* the unique way of carrying out complete determination of the six parameters. Even in the case where only the "golden channel",

$P(\nu_e \to \nu_\mu)$ and $P(\bar{\nu}_e \to \bar{\nu}_\mu)$, is available, one can in principle determine $|\Theta_\pm|^2$, $|\Xi|^2$, $\xi - \theta_\pm$, and their antineutrino counterparts by spectrum analysis. It is because the energy and baseline dependences of the coefficients of these quantities in the oscillation probabilities in (6.11) and (6.14) are different with each other. In the $\nu_\mu - \nu_\tau$ sector all the NSI elements cannot be determined by the rate only analysis, and need for the spectrum information is mandatory in this sector.

It appears that one of the most promising ways to carry this out is the two-detector method [51]. It has been applied to the Tokai-to-Kamioka-Korea (T2KK) two-detector complex which receives neutrino beam from J-PARC [42, 52, 53].[14] In the context of neutrino parameter determination in neutrino factory with NSI as well as SI, this method was examined in detail in [31].[15]

7.7 Parameter degeneracy; old and new

7.7.1 NSI-enriched conventional type degeneracy

The parameter degeneracy is the problem of multiple solutions in determination of lepton mixing parameters [33–35]. It is known to be a notorious problem for their precision measurement. See [56, 57] for a global overview of the degeneracy, and [57, 58] for pictorial representation.

We give evidences that the phenomenon has an extension to the system with NSI.[16] Our discussion based on the matter perturbation theory in section 8 indicates that the parameter degeneracy prevails in system with NSI but with new form which involve NSI parameters. Set of equations for observable we have derived in sections 7.3 and 7.4 shows that the sign-Δm_{31}^2 and the θ_{23} octant degeneracies exist because the equations take different form for different mass hierarchies and octant for a given set of observable. It is also very likely that the intrinsic-type degeneracy survives with a NSI-enriched form, as one can see in the bi-probability diagram [34] given in figure 2 of [31].

7.7.2 New type of degeneracy

Here, we present a completely new type of parameter degeneracy which may be called as the "atmospheric-solar variable exchange" degeneracy. We work with the setting of measurement of six channels at a monochromatic energy. First of all, one notices that determination of the neutrino (un-barred) and the antineutrino (over-barred) variables decouples with each other. We discuss only the neutrino variables below because the antineutrino ones is so similar. To simplify the expressions we restrict ourselves to the case of maximal θ_{23}. By combining (7.5), (7.6), and (7.7) it is easy to show that the phase

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[14] Other possibility would be the one called the "on axis wide-band beam approach" which was proposed in a concrete form in the project description for Brookhaven National Laboratory [54]. Precise estimation of the potential in doing spectrum analysis, however, depends upon which kind of detector is chosen and the actual performance of the detector.

[15] See [55] for effects of the systematic errors and optimization of the similar two-detector setting in parameter determination in neutrino factory.

[16] Notice that introduction of NSI parameters leads to a new solution of the solar neutrino problem [59].

variable can be determined as

$$\tan(\xi - \theta_\pm) = \cot|\Delta_{31}| \frac{P_{e\mu} - P_{\mu e}}{P_{\mu e} - P_{e\tau}}. \tag{7.12}$$

where we have used a simplified notation $P_{\alpha\beta} \equiv P(\nu_\alpha \to \nu_\beta)$.

It is easy to show that if the mass hierarchy is known the solution of this equation is unique in the physical region $-\pi \le \xi \le \pi$ and $-\pi \le \theta_\pm \le \pi$. Then, the solutions for $|\Theta_\pm|$ and $|\Xi|$ are given by

$$|\Theta_\pm|^2 = \frac{P_{e\mu} + P_{e\tau}}{8X_\pm} \left[1 \pm \sqrt{1 - \frac{1}{\sin^2(\xi - \theta_\pm)\sin^2\Delta_{31}} \left(\frac{P_{e\mu} - P_{\mu e}}{P_{e\mu} + P_{e\tau}}\right)^2} \right],$$

$$|\Xi|^2 = \frac{P_{e\mu} + P_{e\tau}}{8Z} \left[1 \mp \sqrt{1 - \frac{1}{\sin^2(\xi - \theta_\pm)\sin^2\Delta_{31}} \left(\frac{P_{e\mu} - P_{\mu e}}{P_{e\mu} + P_{e\tau}}\right)^2} \right]. \tag{7.13}$$

Notice that the degeneracy is quite new; It is the solar-atmospheric variable exchange degeneracy. That is, if there is a solution $|\Theta_\pm^{(1)}|$ and $|\Xi^{(1)}|$, then the second solution $|\Theta_\pm^{(2)}| = \sqrt{\frac{Z}{X_\pm}}|\Xi^{(1)}|$ and $|\Xi^{(2)}| = \sqrt{\frac{X_\pm}{Z}}|\Theta_\pm^{(1)}|$ exists. Notice that the new degeneracy does not survive when NSI is switched off where $\xi = \delta$ and $\theta_\pm = 0$. Namely, there is no phase degree of freedom in the atmospheric variable in the limit, while only phase degree of freedom exists in the solar variable.

Now we turn to the sign-Δm^2 degeneracy. At first sight there is no sign-Δm^2 degeneracy because the sign-Δm^2 flipped solution of $\xi - \theta_\pm$ has to satisfy the same equation (7.12) which has no explicit dependence on the sign. Nevertheless, there is indeed a sign-Δm^2 flipped solution. If $\xi^{(1)}$ and $\theta_+^{(1)}$ are the solution to (7.12) then there are another solutions $\xi^{(2)} = \xi^{(1)} \pm \pi$ and $\theta_-^{(2)} = \theta_+^{(1)} \mp \pi$. It means the existence of the sign-flipped solution of Θ and Ξ, which can be another solutions if accompanied by $(\Delta m_{31}^2)^{(2)} = -(\Delta m_{31}^2)^{(1)}$. With these solutions of the phase equation there exist the similar degenerate solutions as in (7.13). Again the sign-Δm^2 degeneracy does not survive in the no NSI limit because of no degrees of freedom of $\theta_\pm$ in the limit. In conclusion we have uncovered new degeneracies of the intrinsic and the sign-Δm^2 flipped type which exist as a consequence of the presence of NSI.

8 Matter perturbation theory with NSI

As a first step toward understanding the degeneracy we examine neutrino oscillation with NSI by matter perturbation theory following the treatment in [42]. It is known [34] that structure of parameter degeneracy is particularly transparent in the region where the matter effect can be treated as a perturbation, as explicitly verified in the analyses in [42, 52]. See [60] for further explanation of this point.

For simplicity, we restrict our discussion to ν_e related appearance measurement in this section. In concordance to these works we consider ν_e and $\bar{\nu}_e$ appearance measurement with conventional muon neutrino beam and its antiparticles.

8.1 Structure of the oscillation probability with NSI in matter perturbation theory

If we restrict ourselves into the first order in a, the matter effect coefficient, the only terms that survive are the ones up to first order in $\varepsilon_{e\mu}$ or $\varepsilon_{e\tau}$. The oscillation probability in $\nu_\mu \to \nu_e$ channel is given to first order in matter perturbation theory as

$$P(\nu_\mu \to \nu_e; \varepsilon_{e\tau}, \varepsilon_{e\mu}) = P(\nu_\mu \to \nu_e; \varepsilon = 0)_{\text{AKS}}$$
$$+ P(\nu_\mu \to \nu_e; \varepsilon_{e\tau})_{\text{NSI}} + P(\nu_\mu \to \nu_e; \varepsilon_{e\mu})_{\text{NSI}}, \quad (8.1)$$

where the leading term is the Arafune-Koike-Sato (AKS) formula without NSI [18][17]

$$P(\nu_\mu \to \nu_e; \varepsilon = 0)_{\text{AKS}} = \sin^2 2\theta_{13} s_{23}^2 \sin^2 \Delta_{31} + c_{23}^2 \sin^2 2\theta_{12} \left(\frac{\Delta m_{21}^2}{\Delta m_{31}^2}\right)^2 \Delta_{31}^2$$
$$+ 4J_r \left(\frac{\Delta m_{21}^2}{\Delta m_{31}^2}\right) \Delta_{31} \left[\cos\delta \sin 2\Delta_{31} - 2\sin\delta \sin^2 \Delta_{31}\right]$$
$$+ 2\sin^2 2\theta_{13} s_{23}^2 \left(\frac{aL}{4E}\right) \left[\frac{1}{\Delta_{31}} \sin^2 \Delta_{31} - \frac{1}{2} \sin 2\Delta_{31}\right]. \quad (8.2)$$

In (8.2), $\Delta_{31} \equiv \frac{\Delta m_{31}^2 L}{4E}$ $a \equiv 2\sqrt{2}G_F N_e E$ as before. J_r ($\equiv c_{12}s_{12}c_{13}^2 s_{13}c_{23}s_{23}$) denotes the reduced Jarlskog factor.

The first order matter corrections which include the first order NSI effects in ε's can be obtained by taking the first order term in a as

$$P(\nu_\mu \to \nu_e; \varepsilon_{e\tau})_{\text{NSI}} = 8 \left(\frac{aL}{4E}\right)$$
$$\times \left[c_{23} s_{23}^2 s_{13} \left\{|\varepsilon_{e\tau}| \cos(\delta + \phi_{e\tau}) \left(\frac{\sin^2 \Delta_{31}}{\Delta_{31}} - \frac{1}{2} \sin 2\Delta_{31}\right) + |\varepsilon_{e\tau}| \sin(\delta + \phi_{e\tau}) \sin^2 \Delta_{31}\right\}\right.$$
$$\left. - c_{12}s_{12}c_{23}^2 s_{23} \frac{\Delta m_{21}^2}{\Delta m_{31}^2} \left\{|\varepsilon_{e\tau}| \cos\phi_{e\tau} \left(\Delta_{31} - \frac{1}{2}\sin 2\Delta_{31}\right) - |\varepsilon_{e\tau}| \sin\phi_{e\tau} \sin^2 \Delta_{31}\right\}\right], \quad (8.3)$$

$$P(\nu_\mu \to \nu_e; \varepsilon_{e\mu})_{\text{NSI}} = -8 \left(\frac{aL}{4E}\right) \quad (8.4)$$
$$\times \left[s_{23} s_{13} \left\{|\varepsilon_{e\mu}| \cos(\delta + \phi_{e\mu}) \left(s_{23}^2 \frac{\sin^2 \Delta_{31}}{\Delta_{31}} - \frac{c_{23}^2}{2} \sin 2\Delta_{31}\right) + c_{23}^2 |\varepsilon_{e\mu}| \sin(\delta + \phi_{e\mu}) \sin^2 \Delta_{31}\right\}\right.$$
$$\left. - c_{12}s_{12}c_{23} \frac{\Delta m_{21}^2}{\Delta m_{31}^2} \left\{|\varepsilon_{e\mu}| \cos\phi_{e\mu} \left(c_{23}^2 \Delta_{31} + \frac{s_{23}^2}{2} \sin 2\Delta_{31}\right) + s_{23}^2 |\varepsilon_{e\mu}| \sin\phi_{e\mu} \sin^2 \Delta_{31}\right\}\right].$$

The antineutrino probability $P(\bar{\nu}_\mu \to \bar{\nu}_e; \varepsilon_{e\tau}, \varepsilon_{e\mu})$ can be obtained by making the replacement in (8.1); $a \to -a$, $\delta \to 2\pi - \delta$. $\varepsilon_{\alpha\beta} \to \varepsilon_{\alpha\beta}^*$. Notice that both of the CP violating leptonic KM phase δ and $\phi_{\alpha\beta}$ due to NSI elements changes sign when we discuss the time reversal process $\nu_\mu \to \nu_e$, as opposed to $\nu_e \to \nu_\mu$ in the previous sections.

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[17] We got rid of a higher order ϵ^3 term which was kept in our previous references, e.g., [34, 42, 52].

For the oscillation probabilities in the $\nu_\mu - \nu_\tau$ sector we only deal with the one in ν_μ disappearance channel (which may be easiest to measure) to first order in ϵ:

$$P(\nu_\mu \to \nu_\mu; \text{1st order in } \epsilon) \qquad (8.5)$$
$$= 1 - 4c_{23}^2 s_{23}^2 \sin^2 \Delta_{31} + 4c_{12}^2 c_{23}^2 s_{23}^2 \left(\frac{\Delta m_{21}^2}{\Delta m_{31}^2}\right) \Delta_{31} \sin 2\Delta_{31}$$
$$+ 2c_{23}^2 s_{23}^2 \left[(c_{23}^2 - s_{23}^2)(\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}) - 4c_{23}s_{23}\mathrm{Re}(\varepsilon_{\mu\tau})\right] \frac{aL}{2E} \sin 2\Delta_{31}$$
$$- 8c_{23}s_{23}(c_{23}^2 - s_{23}^2) \left[c_{23}s_{23}(\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}) + (c_{23}^2 - s_{23}^2)\mathrm{Re}(\varepsilon_{\mu\tau})\right] \frac{a}{\Delta m_{31}^2} \sin^2 \Delta_{31}.$$

Notice that (8.5) is already in the form of first-order formula in matter perturbation theory.

8.2 Sign-Δm^2 and θ_{23} octant degeneracies prevail in the presence of NSI

In this subsection, we discuss the fate of the sign-Δm^2 and the θ_{23} octant degeneracies in the presence of NSI. In the conventional cases without NSI, they are known as notorious ones among the three types of degeneracies because they are hard to resolve and the former can confuse CP violation with CP conservation. The sign-Δm^2 degeneracy was uncovered in systems without NSI by noticing that the oscillation probability $P(\nu_\mu \to \nu_e)$ in vacuum is invariant under the transformation $\Delta m_{31}^2 \to -\Delta m_{31}^2$, $\delta \to \pi - \delta$ without changing θ_{13} [34]. It maps a positive Δm_{31}^2 solution to the negative one, and vice versa. The presence of the symmetry as well as the fact that it is broken by the first order matter terms can be seen in (8.2).

Now, we observe that the sign-Δm^2 degeneracy prevails in the presence of NSI. That is, the NSI induced terms in the probability (8.3) and (8.4), though they are "matter terms", are invariant under the extended transformation

$$\Delta m_{31}^2 \to -\Delta m_{31}^2,$$
$$\delta \to \pi - \delta,$$
$$\phi_{e\alpha} \to 2\pi - \phi_{e\alpha}. \qquad (8.6)$$

while keeping θ_{13} and $|\varepsilon_{e\alpha}|$ fixed, where $\alpha = \mu, \tau$.[18] The symmetry is broken only by the matter term in (8.2) which is independent of NSI; The symmetry is broken by the matter effect which has exactly the same magnitude in systems with and without NSI. Therefore, to first order in matter perturbation theory, the sign-Δm^2 degeneracy exists in systems with NSI to the same extent as it does in the system without NSI. Given the robustness of the sign-Δm^2 degeneracy in the conventional case we suspect that the degeneracy in systems with NSI has the similar robustness.

Similarly, one can easily show that the θ_{23} octant degeneracy survives the presence of NSI. It can be readily observed that $P(\nu_\mu \to \nu_\mu; \text{1st order in } \epsilon)$ in (8.5) is invariant under

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[18] Under the transformation (8.6), the trigonometric factors in (8.4) and (8.3) transform as follows: $\cos(\delta + \phi_{e\alpha}) \to -\cos(\delta + \phi_{e\alpha})$, $\sin(\delta + \phi_{e\alpha}) \to +\sin(\delta + \phi_{e\alpha})$, $\cos\phi_{e\alpha} \to +\cos\phi_{e\alpha}$, and $\sin\phi_{e\alpha} \to -\sin\phi_{e\alpha}$.

the transformation

$$c_{23} \to s_{23},$$
$$s_{23} \to c_{23},$$
$$(\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}) \to -(\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}). \tag{8.7}$$

It means that the θ_{23} octant degeneracy prevails in the presence of NSI, and actually in an extended form which involves NSI parameter $\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}$. Since this NSI parameter decouples from $P(\nu_\mu \to \nu_e)$ to second-order in ϵ, the presence of the θ_{23} octant degeneracy remains intact when the NSI is included though values of the degenerate solutions themselves are affected by the presence of $\varepsilon_{e\alpha}$.

It is interesting to note that both of the two degeneracies discussed in this subsection have common features. Their presence can be discussed based on (approximate) invariance under some discrete transformations, and with NSI the transformations are extended to the ones which involve NSI parameters. Most probably, our treatment here is the first one to signal the existence of the degenerate solutions which involves both the SI (θ_{13} and δ) and the NSI parameters.

8.3 Decoupling between the degeneracies in the presence of NSI

In [42] the property called "decoupling between degeneracies" are shown to exist for experimental settings with baseline shorter than ~ 1000 km which may allow treatment based on matter perturbation theory. See also [39] and [61] for preliminary discussions. The property of decoupling between degeneracies A and B guarantees that when one tries to resolve the degeneracy A one can forget about the presence of the degeneracy B, and vice versa. Existence of NSI terms, in general, influences the discussion of decoupling. It is the purpose of this and the next subsections to fully discuss the fate of the decoupling in the presence of NSI. Since it is one of the most significant characteristic features of the degeneracies in matter perturbative regime, we believe it worth to present a complete treatment.

8.3.1 Definition of decoupling between degeneracies

To define the concept of decoupling between degeneracies A and B, we introduce, following [42], the probability difference

$$\Delta P^{ab}(\nu_\alpha \to \nu_\beta) \equiv P\left(\nu_\alpha \to \nu_\beta; (\Delta m_{31}^2)^{(a)}, \theta_{23}^{(a)}, \theta_{13}^{(a)}, \delta^{(a)}, \varepsilon_{\alpha\beta}^{(a)}\right)$$
$$- P\left(\nu_\alpha \to \nu_\beta; (\Delta m_{31}^2)^{(b)}, \theta_{23}^{(b)}, \theta_{13}^{(b)}, \delta^{(b)}, \varepsilon_{\alpha\beta}^{(b)}\right), \tag{8.8}$$

where the superscripts a and b label the degenerate solutions. Suppose that we are discussing the degeneracy A. The decoupling between the degeneracies A and B holds if ΔP^{ab} defined in (8.8) for the degeneracy A is invariant under the replacement of the mixing parameters corresponding to the degeneracy B, and vice versa.

8.3.2 Matter-perturbative treatment of the degenerate solutions

We follow [42] to define the degenerate solutions in a perturbative manner.[19] Throughout the discussion in this section we assume that deviation of θ_{23} from the maximal angle $\pi/4$ is small. A disappearance measurement, $\nu_\mu \to \nu_\mu$, determines s_{23}^2 to first order in s_{13}^2 as $(s_{23}^2)^{(1)} = (s_{23}^2)^{(0)}(1 + s_{13}^2)$, where $(s_{23}^2)^{(0)}$ is the solution obtained by ignoring s_{13}^2. It is given by $(s_{23}^2)^{(0)} = \frac{1}{2}\left[1 \pm \sqrt{1 - \sin^2 2\theta_{23}}\right]$. In leading order the relationship between the first and the second octant solutions of θ_{23} is given by $s_{23}^{1st} = c_{23}^{2nd}$.

A ν_e appearance measurement determines the combination $s_{23}^2 \sin^2 2\theta_{13}$. The first and the second octant solutions of θ_{23} are also related to leading order by $s_{23}^{1st} s_{13}^{1st} = s_{23}^{2nd} s_{13}^{2nd}$. In an environment where the vacuum oscillation approximation applies the solutions corresponding to the intrinsic degeneracy are given in appendix D as

$$\theta_{13}^{(2)} = \sqrt{(\theta_{13}^{(1)})^2 + 2\left(\frac{Y_c}{X}\right)\theta_{13}^{(1)}\cos\delta_1 + \left(\frac{Y_c}{X}\right)^2}$$

$$\sin\delta_2 = \frac{\theta_{13}^{(1)}}{\theta_{13}^{(2)}}\sin\delta_1$$

$$\cos\delta_2 = \mp\frac{1}{\theta_{13}^{(2)}}\left(\theta_{13}^{(1)}\cos\delta_1 + \frac{Y_c}{X}\right) \tag{8.9}$$

where

$$\frac{Y_c}{X} \equiv \sin 2\theta_{12}\cot\theta_{23}\Delta_{21}\cot\Delta_{31}. \tag{8.10}$$

and the superscripts (1) and (2) label the solutions due to the intrinsic degeneracy. The sign $\mp$ for $\cos\delta_2$ are for $Y_c = \pm|Y_c|$, and $\theta_{13}^{(2)}$ in the solution of δ is meant to be the $\theta_{13}^{(2)}$ solution given in the first line in (8.9).

As we saw in the previous section, an extended form of the sign-Δm^2 degeneracy is given under the same approximation (mod. 2π) as

$$\theta_{13}^{\rm norm} = \theta_{13}^{\rm inv}, \quad (\Delta m_{31}^2)^{\rm norm} = -(\Delta m_{31}^2)^{\rm inv}, \quad \delta^{\rm norm} = \pi - \delta^{\rm inv}, \quad (\phi_{\alpha\beta})^{\rm norm} = -(\phi_{\alpha\beta})^{\rm inv}, \tag{8.11}$$

where the superscripts "norm" and "inv" label the solutions with the positive and the negative sign of Δm_{31}^2, and $\phi_{\alpha\beta}$ denotes the phase of $\varepsilon_{\alpha\beta}$. The validity of these approximate relationships in the actual experimental setup in the T2K II measurement is explicitly verified in [42, 52]. It should be noticed that even if sizable matter effect is present the relation (8.11) holds in a good approximation if the energy is tuned to the one corresponding to the vacuum oscillation maximum, or more precisely, the shrunk ellipse limit [62].

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[19] More precise meaning of the term "perturbative" is as follows: Since the disappearance probability by which θ_{23} is determined is of order unity we disregard quantities of order ϵ or higher. They include the matter effect, θ_{13}, and NSI. Similarly, ν_e appearance probability is of order ϵ^2 the relationship between the two degenerate solution inevitably contains a small quantity, which is θ_{13} in this case. But, all the quantities of higher order are neglected. If the near-far two detectors are involved, like in the case of T2KK [42, 52], the degenerate solutions are essentially defined by the near detector. In this case, the second detector is meant to give raise to perturbation effect to lift the degeneracy. For more concrete example of this feature, see [42].

8.4 Decoupling between the sign-Δm^2 and the θ_{23} octant degeneracies

Let us start by treating the sign-Δm^2 degeneracy. For this purpose, we calculate $\Delta P^{\text{norm inv}}(\nu_\mu \to \nu_e)$ as defined in (8.8). Thanks to the extended symmetry (8.6) obeyed by the appearance probability, it is given by the same result obtained without NSI in [42]:

$$\Delta P^{\text{norm inv}}(\nu_\mu \to \nu_e) = \tag{8.12}$$
$$\sin^2 2\theta_{13}^{\text{norm}} (s_{23}^{\text{norm}})^2 \left(\frac{aL}{E}\right) \left[\frac{1}{(\Delta_{31})^{\text{norm}}} \sin^2(\Delta_{31})^{\text{norm}} - \frac{1}{2} \sin 2(\Delta_{31})^{\text{norm}}\right]$$

where the superscripts "norm" and "inv" can be exchanged if one want to start from the inverted hierarchy. Therefore, breaking the sign-Δm^2 degeneracy requires the matter effect but not more than that required in resolving it in systems without NSI; NSI does not contribute resolution of the sign-Δm^2 degeneracy but it does not add more difficulties.

By following the same discussion as in [42], we observe that $\Delta P^{\text{norm inv}}$ is invariant under the transformation $\theta_{23}^{\text{1st}} \leftrightarrow \theta_{23}^{\text{2nd}}$ and $\theta_{13}^{\text{1st}} \leftrightarrow \theta_{13}^{\text{2nd}}$, because $\Delta P^{\text{norm inv}}$ depends upon θ_{13} and θ_{23} only through the combination $\sin^2 2\theta_{13} s_{23}^2$ within our approximation. Therefore, resolution of the sign-Δm_{31}^2 can be done in the presence of the θ_{23} octant degeneracy.

What is the influence of the ν_μ disappearance channel in the discussion of decoupling? Using the first-order formula in (8.5), $\Delta P^{\text{norm inv}}(\nu_\mu \to \nu_\mu)$ can be computed as

$$\Delta P^{\text{norm inv}}(\nu_\mu \to \nu_\mu) = 8 c_{12}^2 c_{23}^2 s_{23}^2 \left(\frac{\Delta m_{21}^2 L}{4E}\right) \sin\frac{\Delta m_{31}^2 L}{2E}$$
$$+ 4 c_{23}^2 s_{23}^2 \left[(c_{23}^2 - s_{23}^2)(\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}) - 4 c_{23} s_{23} \text{Re}(\varepsilon_{\mu\tau})\right] \frac{aL}{2E} \sin\frac{\Delta m_{31}^2 L}{2E}. \tag{8.13}$$

It is manifestly invariant under that the transformation in (8.7), and hence the sign-Δm_{31}^2 degeneracy decouples from the θ_{23} octant degeneracy. Presence of the $\Delta P^{\text{norm inv}}(\nu_\mu \to \nu_\mu)$ in first order in ϵ indicates that the ν_μ disappearance channel would play a role in lifting the sign-Δm_{31}^2 degeneracy if the measurement is done off the vacuum oscillation maximum.

Now, we discuss the inverse problem, namely, whether the θ_{23} octant degeneracy can be resolved in the presence of the sign-Δm_{31}^2 degeneracy. By noting that $J_r^{\text{1st}} - J_r^{\text{2nd}} = \cos 2\theta_{23}^{\text{1st}} J_r^{\text{1st}}$ in leading order in $\cos 2\theta_{23}$, the difference between probabilities with the first and the second octant solutions can be given by

$$\Delta P^{\text{1st 2nd}}(\nu_\mu \to \nu_e) = \cos 2\theta_{23}^{\text{1st}} \Delta_{21} \left[\sin^2 2\theta_{12} \Delta_{21} + 4 J_r^{\text{1st}} \left(\cos\delta \sin 2\Delta_{31} - 2\sin\delta \sin^2 \Delta_{31}\right)\right]$$
$$+ \Delta P^{\text{1st 2nd}}(\nu_\mu \to \nu_e; \varepsilon_{e\tau})_{\text{NSI}} + \Delta P^{\text{1st 2nd}}(\nu_\mu \to \nu_e; \varepsilon_{e\mu})_{\text{NSI}}, \tag{8.14}$$

where

$$\Delta P^{\text{1st 2nd}}(\nu_\mu \to \nu_e; \varepsilon_{e\tau})_{\text{NSI}} = = -2\sqrt{2} c_{12} s_{12} \cos 2\theta_{23}^{\text{1st}} \sin 2\theta_{23}^{\text{1st}} \left(\frac{aL}{4E}\right) \left(\frac{\Delta m_{21}^2}{\Delta m_{31}^2}\right) \tag{8.15}$$
$$\times \left[|\varepsilon_{e\tau}| \cos\phi_{e\tau} \left(\Delta_{31} - \frac{1}{2}\sin 2\Delta_{31}\right) - |\varepsilon_{e\tau}| \sin\phi_{e\tau} \sin^2 \Delta_{31}\right],$$

$$\Delta P^{\text{1st 2nd}}(\nu_\mu \to \nu_e; \varepsilon_{e\mu})_{\text{NSI}} = 8 \cos 2\theta_{23}^{\text{1st}} \left(\frac{aL}{4E} \right) \tag{8.16}$$

$$\times \left[s_{23}^{\text{1st}} s_{13} \left\{ |\varepsilon_{e\mu}| \cos(\delta + \phi_{e\mu}) \left(\frac{\sin^2 \Delta_{31}}{\Delta_{31}} + \frac{1}{2} \sin 2\Delta_{31} \right) - |\varepsilon_{e\mu}| \sin(\delta + \phi_{e\mu}) \sin^2 \Delta_{31} \right\} \right.$$

$$\left. + \frac{c_{12} s_{12}}{2\sqrt{2}} \left(\frac{\Delta m_{21}^2}{\Delta m_{31}^2} \right) \left\{ |\varepsilon_{e\mu}| \cos \phi_{e\mu} \left(3\Delta_{31} - \frac{1}{2} \sin 2\theta_{23}^{\text{1st}} \sin 2\Delta_{31} \right) - |\varepsilon_{e\mu}| \sin \phi_{e\mu} \sin 2\theta_{23}^{\text{1st}} \sin^2 \Delta_{31} \right\} \right].$$

The first term of $\Delta P^{\text{1st 2nd}}$ in (8.14), being composed only of the vacuum oscillation terms, is obviously invariant under the replacement *normal $\leftrightarrow$ inverted* solutions. The remarkable feature of (8.15) and (8.16) is that they are also invariant under the replacement relation between different hierarchy solutions given in (8.6) which is extended to include NSI phases. The disappearance channel does not play a role in the present discussion under the approximation taken in deriving (8.5), because then $\Delta P^{\text{1st 2nd}}(\nu_\mu \to \nu_\mu)$ vanishes. Therefore, even in the presence of NSI, the resolution of the θ_{23} octant degeneracy can be carried out without worrying about the presence of the sign-Δm_{31}^2 degeneracy. The sign-Δm^2 and the θ_{23} octant degeneracies decouple with each other even in the presence of NSI in matter perturbative regime.

8.5 Non-decoupling of intrinsic degeneracy

Now we discuss the intrinsic degeneracy for which the situation is somewhat different. First of all, this is the degeneracy which is somewhat different in nature. Unlike the case of the sign-Δm_{31}^2 degeneracy, this degeneracy is known to be fragile to the spectrum analysis; In many cases it can be resolved by including informations of energy dependence in the reconstructed events. An example for this is the T2KK setting which receives an intense neutrino beam from J-PARC [42, 52]. It means that in this case there is no intrinsic degeneracy from the beginning. Nonetheless, anticipating possible circumstances in which spectrum informations are not available, and for completeness, we discuss below if resolving the intrinsic degeneracy decouple to lifting the other two degeneracies. We disregard the ν_μ disappearance channel in this subsection because it does not appear to play a major role in resolving the intrinsic degeneracy. The discussions in this subsection are also meant to partly correct and append the ones given in section III in [42].

8.5.1 Non-decoupling of intrinsic degeneracy without NSI

Let us first discuss the problem of decoupling with intrinsic degeneracy without NSI. In our perturbative approach $\Delta P^{12}(\nu_\mu \to \nu_e)$ arises only from the first order matter term in (8.2) because the degenerate solutions in vacuum, by definition, gives the same vacuum oscillation probabilities. It reads

$$\Delta P^{12}(\nu_\mu \to \nu_e) = -4 \left(\frac{aL}{4E} \right) \Delta \theta^2 \left(\frac{1}{\Delta_{31}} \sin^2 \Delta_{31} - \frac{1}{2} \sin 2\Delta_{31} \right). \tag{8.17}$$

where

$$\Delta \theta^2 \equiv (\theta_{13}^{(1)})^2 - (\theta_{13}^{(2)})^2 \tag{8.18}$$

$$= -\sin 2\theta_{12} \cot \theta_{23} \Delta_{21} \cot \Delta_{31} \left(2\theta_{13}^{(1)} \cos \delta^{(1)} + \sin 2\theta_{12} \cot \theta_{23} \Delta_{21} \cot \Delta_{31} \right)$$

Based on the result of ΔP^{12} in (8.17) we discuss possible decoupling of the sign-Δm^2 and the octant θ_{23} degeneracies from the intrinsic one.

We start from the sign-Δm^2 degeneracy. It can be readily seen that ΔP^{12} is odd under interchange of the normal and the inverted hierarchy solutions as dictated in (8.11). It means that $\Delta P^{12}(normal) - \Delta P^{12}(inverted) = 2\Delta P^{12}$. Clearly, the sign-$\Delta m^2$ degeneracy do not decouple from the intrinsic one.

Now we turn to the octant θ_{23} degeneracy. From (8.17), $\Delta P^{12}(\text{1st}) - \Delta P^{12}(\text{2nd})$ reads

$$
\Delta P^{12}(\text{1st}) - \Delta P^{12}(\text{2nd}) = 8\left(\frac{aL}{4E}\right) \sin 2\theta_{12}\Delta_{21} \cot\Delta_{31} \left(\frac{1}{\Delta_{31}} \sin^2\Delta_{31} - \frac{1}{2}\sin 2\Delta_{31}\right)
$$
$$
\times \cos 2\theta_{23}\left[\theta_{13}^{(1)}\cos\delta^{(1)} \frac{1 + c_{23}s_{23}}{c_{23}^2 s_{23}(c_{23}+s_{23})} + \frac{2}{\sin^2 2\theta_{23}}\sin 2\theta_{12}\Delta_{21}\cot\Delta_{31}\right] \quad (8.19)
$$

where θ_{13} and s_{23} etc. in (8.19) are meant to be the ones in the first octant. It is small in the sense that it is proportional to $\cos 2\theta_{23}$ which vanishes in the limit of maximal θ_{23}. But, this is the factor of kinematical origin which inevitably exists because the measure for breaking of the octant degeneracy has to vanish at $\theta_{23} = \pi/4$. Therefore, we conclude that there is no dynamical decoupling of the θ_{23} octant degeneracy from the intrinsic one.

Now, we discuss the inverse problem, namely, whether the sign-Δm^2 and the θ_{23} octant degeneracies can be resolved independently of the intrinsic degeneracy. The measure for resolving the sign-Δm^2 degeneracy is given in (8.12)

$$
\Delta P^{\text{norm inv}}(1) - \Delta P^{\text{norm inv}}(2) = 4\Delta\theta^2 s_{23}^2 \left(\frac{aL}{E}\right)\left[\frac{1}{\Delta_{31}}\sin^2\Delta_{31} - \frac{1}{2}\sin 2\Delta_{31}\right]\Big|_{\text{norm}}^{(1)} \quad (8.20)
$$

where all the quantities in (8.20) is to be evaluated by using the normal hierarchy and intrinsic first solution. Clearly, the intrinsic degeneracy does not decouple from the sign-Δm^2 one.

How about the θ_{23} octant degeneracy? The appropriate measure for the question is given by

$$
\Delta P^{\text{1st 2nd}}(1) - \Delta P^{\text{1st 2nd}}(2) = \cos 2\theta_{23}^{\text{1st}}\Delta_{21} \quad (8.21)
$$
$$
\times \frac{1}{\theta_{13}^{(2)}}\left[\left\{4J_r^{\text{1st}}\left(\theta_{13}^{(1)} + \theta_{13}^{(2)}\right)\cos\delta^{(1)} + \frac{Y_c}{X}\right\}\sin 2\Delta_{31} - 2\frac{\sin\delta^{(1)}}{\theta_{13}^{(1)} + \theta_{13}^{(2)}}\Delta\theta^2\sin^2\Delta_{31}\right]
$$

where $\theta_{13}^{(2)}$ implies to insert the expression in (8.9). Again there is no sign of the decoupling.

Nonetheless, there are some cases in which the decoupling with the intrinsic degeneracy still holds in a good approximation. For example, $\Delta P^{12}(\text{1st}) - \Delta P^{12}(\text{2nd})$ in (8.19) and $\Delta P^{\text{norm inv}}(1) - \Delta P^{\text{norm inv}}$ in (8.20) may be small numerically. It is the case at relatively short baseline $L \lesssim 1000$ km where it is further suppressed by $\frac{aL}{4E}$. The ΔP differences between the two θ_{23} octant solutions are always suppressed by $\cos 2\theta_{23}$, and hence they may be small at θ_{23} very close to the maximal.

It is significant to observe that at the vacuum oscillation maxima, $\Delta_{31} = (2n+1)\frac{\pi}{2}$, the decoupling is realized in all pairs of degeneracies. Therefore, if the experimental set up is near the vacuum oscillation maxima the decoupling with the intrinsic degeneracy

perfectly holds. The identical two detector setting in T2KK [42, 52], whose intermediate (far) detector is near the first (second) oscillation maximum provides a good example for such "accidental decoupling".

8.5.2 Decoupling and non-decoupling of intrinsic degeneracy with NSI

We concisely describe what happens in the decoupling between the intrinsic and the other two degeneracies when NSI is introduced. We explicitly discuss below the case with $\varepsilon_{e\tau}$ because the equations are slightly simpler, but we have verified that the same conclusion holds for the case with $\varepsilon_{e\mu}$, and hence in the full system.

$\varepsilon_{e\tau}$ type NSI gives rise to contribution to the difference of the probabilities with the first and the second solutions of intrinsic degeneracy of the following form

$$\Delta P^{12}(\nu_\mu \to \nu_e; \varepsilon_{e\tau}) = 8 \left(\frac{aL}{4E}\right) |\varepsilon_{e\tau}| c_{23} s_{23}^2 \tag{8.22}$$
$$\times \left(2\theta_{13}^{(1)} \cos \delta^{(1)} + \frac{Y_c}{X}\right) \left[\cos \phi_{e\tau} \left(\frac{\sin^2 \Delta_{31}}{\Delta_{31}} - \frac{1}{2} \sin 2\Delta_{31}\right) + \sin \phi_{e\tau} \sin^2 \Delta_{31}\right],$$

where use has been made of the relation (D.7). Notice that the terms proportional to the solar Δm_{21}^2 do not contribute, and $\sin \delta$ terms cancel out owing to the relation (D.6).

We observe that $\Delta P^{12}(\varepsilon_{e\tau})$ are invariant under interchange between the normal and the inverted hierarchies, (8.11). Therefore, NSI induced oscillation probability, by itself, fulfills the decoupling condition with the sign-Δm^2 degeneracy.

The situation is different in relationship with the θ_{23} octant degeneracy. With $\varepsilon_{e\tau}$ one can derive the similar expression as (8.19):

$$\Delta P^{12}(\varepsilon_{e\tau}; 1\text{st}) - \Delta P^{12}(\varepsilon_{e\tau}; 2\text{nd}) = 4\sqrt{2} \cos 2\theta_{23} \left(\frac{aL}{4E}\right) |\varepsilon_{e\tau}| \sin 2\theta_{12} \Delta_{21} \cot \Delta_{31}$$
$$\times \left[\cos \phi_{e\tau} \left(\frac{\sin^2 \Delta_{31}}{\Delta_{31}} - \frac{1}{2} \sin 2\Delta_{31}\right) + \sin \phi_{e\tau} \sin^2 \Delta_{31}\right]. \tag{8.23}$$

Though the intrinsic degeneracy does not decouple with the θ_{23} octant degeneracy, the suppression factor $\cos 2\theta_{23} \left(\frac{aL}{4E}\right) |\varepsilon_{e\tau}|$ may be very small if baseline is relatively short and θ_{23} is near maximal, assuming the likely possibility that $|\varepsilon_{e\tau}|$ is small. Again, the decoupling holds at the vacuum oscillation maxima.

General conclusion in the last two subsections is that although the decoupling between the sign-Δm_{31}^2 and the θ_{23} octant degeneracies holds, there is no decoupling between the intrinsic degeneracy and the other two types of degeneracies. The conclusion applies to the cases with and without NSI.

9 Concluding remarks

In this paper, we have discussed various aspects of neutrino oscillation with NSI, the exactly hold properties as well as the properties best illuminated by a perturbative method. The former category includes the relation between the S matrix elements and the probabilities that arises due to an invariance of the Hamiltonian under the transformation (4.13) which involves θ_{23} and the NSI elements $\varepsilon_{\alpha\beta}$ $(\alpha, \beta = e, \mu, \tau)$. It allows us to connect the

probabilities of various flavor conversion channels, which is powerful enough to strongly constrain the way how various NSI elements $\varepsilon_{\alpha\beta}$ enter into the oscillation probabilities. This category also includes the phase reduction theorem which guarantees reduction of number of CP violating phases when the solar Δm^2_{21} is switched off.

By taking the following three quantities, $\frac{\Delta m^2_{21}}{\Delta m^2_{31}}$, s_{13}, and the NSI elements $\varepsilon_{\alpha\beta}$, as small expansion parameters (which are collectively denoted as ϵ) we have formulated a perturbative framework which we have dubbed as the "ϵ perturbation theory". Within this framework we have calculated the S matrix elements to order ϵ^2 and derived the NSI second-order formula of the oscillation probability in all channels. It allows us to estimate size of the contribution of the particular NSI element $\varepsilon_{\alpha\beta}$ ($\alpha, \beta = e, \mu, \tau$) to the particular oscillation probability $P(\nu_\kappa \to \nu_\omega)$ ($\kappa, \omega = e, \mu, \tau$), as tabulated in table 1. To complete the table (and for other reasons) we have also calculated the oscillation probability in the ν_e related channels to third order in ϵ, which is given in appendix C. We have given a global overview of neutrino oscillation with NSI and hope that the table serves as a "handbook" for hunting NSI effects in neutrino propagation.

Thanks to the NSI second-order formula we have discussed, for the first time, the way how the SI and the NSI parameters can be determined simultaneously. We found that measurement of all the relevant NSI and SI parameters is extremely demanding; While all the NSI elements in ν_e related sector can in principle be determined together with θ_{13} and δ, it requires $\nu_e \to \nu_\mu$, $\nu_\mu \to \nu_e$, $\nu_e \to \nu_\tau$, and their CP conjugate channels if we do it by the rate only measurement. We have also proven to the accuracy of ϵ^2 that, if we restrict to the rate only analysis, all the NSI elements in $\nu_\mu - \nu_\tau$ sector cannot be determined even if we prepare ν_τ beam.

Clearly, the right strategy is to pursue the appropriate experimental setup which enables us the spectrum analysis to determine several coefficients at the same time. The capability of spectrum analysis with good resolution would be a mandatory requirement for future facilities which aim at searching for effects of NSI at least as one of their objectives. To our knowledge, the leading candidate for such setup is the two-detector setup at $L \simeq 3000$ km and $L \simeq 7000$ km in neutrino factory with use of the golden channel [20], which are proven to be powerful in resolving the conventional parameter degeneracy [33, 63]. In a previous paper, it was shown that the setting is also powerful in resolving the θ_{13}-NSI (and probably the two-phase) confusion [31]. It must be stressed, however, that we still do not know if the setting is sufficiently powerful in determining all the SI and the NSI parameters.

We have observed that the phenomenon of parameter degeneracy prevails in the system with NSI. Notably, it exists in an extended form of involving not only the SI but also the NSI parameters. In a concrete setting of six probabilities at monochromatic beam, we have uncovered a new type of degeneracy, the solar-atmospheric variable exchange degeneracy. To have a first grasp of the nature of the parameter degeneracy of more conventional type, we have discussed the matter perturbation theory of neutrino oscillation with NSI. We have found that the sign-Δm^2_{31} and the θ_{23} octant degeneracies are robust, and the analysis indicates the way how the NSI parameters are involved into the new form of degeneracy. The decoupling between degeneracies, a salient feature in the matter perturbative regime, is also revisited in an extended setting with NSI.

In our investigation we have also noticed a new feature of neutrino oscillation in matter in the standard three-flavor oscillation without NSI, that is, the matter hesitation. It states that the matter effect comes in into the oscillation probability only at the second order in ϵ. The property allows us to understand why it is so difficult to detect the matter effect in various long-baseline experiments, and explains why ε_{ee} is absent from the NSI second order formula. Notice that the property does not hold in the $\nu_\mu - \nu_\tau$ system with NSI.

Of course, a number of cautions have to be made to correctly interpret our results; Many of our statements are based on the NSI second order formula which is reliable only if the assumptions made in formulating our perturbative treatment are correct. We do not deal with effects of NSI in production and detection of neutrinos. The program of complete determination of the NSI parameters mentioned above must be cooperated with search for NSI in production and detection processes.

In this paper we confined the case of relatively small θ_{13} in accordance to our perturbative hypothesis in (5.1). What happens if θ_{13} is large enough so that not only θ_{13} but also δ are determined by the next generation reactor/accelerator [64–66] and upgraded superbeam [67] experiments prior NSI search? Then, one might argue that the discussion of parameter determination would become much less complicated in this case. We argue that this is not quite correct. As we have seen in section 6 the NSI and the SI parameters appear in the oscillation probability in a tightly coupled way. Hence, determination of the former with size of $\varepsilon_{\alpha,\beta} \sim 10^{-2}$ requires simultaneous determination of the latter with *accuracy* of the similar order. Therefore, prior determination of θ_{13} and δ, unless extremely precise ones, would not alter the necessity of simultaneous determination of SI and NSI parameters. However, we note that knowing the neutrino mass hierarchy would greatly help by decreasing the ambiguities which arise from the degeneracy.

A S matrix elements for neutrino oscillation with NSI

Using the formalism described in section 5 with the double-tilde basis (6.1) it is straightforward to compute the S matrix elements for neutrino oscillations with NSI. Omitting calculations we just present the results of the S matrix elements: The notations used below are: $\Delta \equiv \frac{\Delta m_{31}^2}{2E}$, $r_\Delta \equiv \frac{\Delta m_{21}^2}{\Delta m_{31}^2}$, $r_A \equiv \frac{a}{\Delta m_{31}^2}$, and the NSI elements are in the tilde-basis (4.12).

$$
\begin{aligned}
S_{ee} = & \left\{ 1 - i\Delta L \left(s_{12}^2 r_\Delta + r_A \tilde{\varepsilon}_{ee} \right) \right\} e^{-ir_A \Delta L} \\
& + s_{13}^2 (ir_A \Delta L) e^{-ir_A \Delta L} - s_{13}^2 \frac{1+r_A}{1-r_A} \left(e^{-ir_A \Delta L} - e^{-i\Delta L} \right) \\
& - 2s_{13} \operatorname{Re}(\tilde{\varepsilon}_{e\tau} e^{i\delta}) r_A \left[i\Delta L e^{-ir_A \Delta L} + \frac{1}{1-r_A} \left(e^{-ir_A \Delta L} - e^{-i\Delta L} \right) \right] \\
& - (s_{12}^2 \frac{r_\Delta}{r_A} + \tilde{\varepsilon}_{ee})^2 \frac{(r_A \Delta L)^2}{2} e^{-ir_A \Delta L} \\
& - |c_{12} s_{12} \frac{r_\Delta}{r_A} + \tilde{\varepsilon}_{e\mu}|^2 \left\{ (ir_A \Delta L) e^{-ir_A \Delta L} - \left(1 - e^{-ir_A \Delta L} \right) \right\} \\
& + |s_{13} e^{-i\delta} + \tilde{\varepsilon}_{e\tau}|^2 \left(\frac{r_A^2}{1-r_A} \right) \left[i\Delta L e^{-ir_A \Delta L} - \frac{1}{1-r_A} \left(e^{-ir_A \Delta L} - e^{-i\Delta L} \right) \right]
\end{aligned}
\tag{A.1}
$$

$$S_{e\mu} = -c_{23}\left(c_{12}s_{12}\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{e\mu}\right)\left(1-e^{-ir_A\Delta L}\right) - s_{23}\left(s_{13}e^{-i\delta}+r_A\tilde{\varepsilon}_{e\tau}\right)\frac{1}{1-r_A}\left(e^{-ir_A\Delta L}-e^{-i\Delta L}\right)$$

$$-c_{23}s_{13}\tilde{\varepsilon}_{\mu\tau}^*e^{-i\delta}\left\{\left(1-e^{-ir_A\Delta L}\right)+r_A\left(1-e^{-i\Delta L}\right)\right\}$$

$$+s_{23}s_{13}e^{-i\delta}(\tilde{\varepsilon}_{ee}-\tilde{\varepsilon}_{\tau\tau})\frac{r_A}{1-r_A}\left(e^{-ir_A\Delta L}-e^{-i\Delta L}\right)$$

$$+s_{23}s_{13}e^{-i\delta}(i\Delta L)\left\{(s_{12}^2 r_\Delta+\tilde{\varepsilon}_{ee}r_A)e^{-ir_A\Delta L}-\tilde{\varepsilon}_{\tau\tau}r_A e^{-i\Delta L}\right\}$$

$$+c_{23}\left(c_{12}s_{12}\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{e\mu}\right)\left[ir_A\Delta L\left\{\left(c_{12}^2\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{\mu\mu}\right)-\left(s_{12}^2\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{ee}\right)e^{-ir_A\Delta L}\right\}\right.$$

$$\left.-\left((c_{12}^2-s_{12}^2)\frac{r_\Delta}{r_A}-\tilde{\varepsilon}_{ee}+\tilde{\varepsilon}_{\mu\mu}\right)\left(1-e^{-ir_A\Delta L}\right)\right]$$

$$+s_{23}\left(s_{13}e^{-i\delta}+\tilde{\varepsilon}_{e\tau}\right)\left(\frac{r_A^2}{1-r_A}\right)\left[i\Delta L\left\{\left(s_{12}^2\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{ee}\right)e^{-ir_A\Delta L}-\tilde{\varepsilon}_{\tau\tau}e^{-i\Delta L}\right\}\right.$$

$$\left.-\frac{1}{1-r_A}\left(s_{12}^2\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{ee}-\tilde{\varepsilon}_{\tau\tau}\right)\left(e^{-ir_A\Delta L}-e^{-i\Delta L}\right)\right]$$

$$+\left\{c_{23}\tilde{\varepsilon}_{\mu\tau}^*\left(s_{13}e^{-i\delta}+\tilde{\varepsilon}_{e\tau}\right)+s_{23}\tilde{\varepsilon}_{\mu\tau}\left(c_{12}s_{12}\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{e\mu}\right)\right\}$$

$$\times r_A\left[\left(1-e^{-i\Delta L}\right)-\frac{1}{1-r_A}\left(e^{-ir_A\Delta L}-e^{-i\Delta L}\right)\right] \tag{A.2}$$

$$S_{e\tau} = s_{23}\left(c_{12}s_{12}\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{e\mu}\right)\left(1-e^{-ir_A\Delta L}\right) - c_{23}\left(s_{13}e^{-i\delta}+r_A\tilde{\varepsilon}_{e\tau}\right)\frac{1}{1-r_A}\left(e^{-ir_A\Delta L}-e^{-i\Delta L}\right)$$

$$+s_{23}s_{13}\tilde{\varepsilon}_{\mu\tau}^*e^{-i\delta}\left\{\left(1-e^{-ir_A\Delta L}\right)+r_A\left(1-e^{-i\Delta L}\right)\right\}$$

$$+c_{23}s_{13}e^{-i\delta}(\tilde{\varepsilon}_{ee}-\tilde{\varepsilon}_{\tau\tau})\frac{r_A}{1-r_A}\left(e^{-ir_A\Delta L}-e^{-i\Delta L}\right)$$

$$+c_{23}s_{13}e^{-i\delta}(i\Delta L)\left\{(s_{12}^2 r_\Delta+\tilde{\varepsilon}_{ee}r_A)e^{-ir_A\Delta L}-\tilde{\varepsilon}_{\tau\tau}r_A e^{-i\Delta L}\right\}$$

$$+s_{23}\left(c_{12}s_{12}\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{e\mu}\right)\left[ir_A\Delta L\left\{-\left(c_{12}^2\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{\mu\mu}\right)+\left(s_{12}^2\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{ee}\right)e^{-ir_A\Delta L}\right\}\right.$$

$$\left.+\left((c_{12}^2-s_{12}^2)\frac{r_\Delta}{r_A}-\tilde{\varepsilon}_{ee}+\tilde{\varepsilon}_{\mu\mu}\right)\left(1-e^{-ir_A\Delta L}\right)\right]$$

$$+c_{23}\left(s_{13}e^{-i\delta}+\tilde{\varepsilon}_{e\tau}\right)\left(\frac{r_A^2}{1-r_A}\right)\left[i\Delta L\left\{\left(s_{12}^2\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{ee}\right)e^{-ir_A\Delta L}-\tilde{\varepsilon}_{\tau\tau}e^{-i\Delta L}\right\}\right.$$

$$\left.-\frac{1}{1-r_A}\left(s_{12}^2\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{ee}-\tilde{\varepsilon}_{\tau\tau}\right)\left(e^{-ir_A\Delta L}-e^{-i\Delta L}\right)\right]$$

$$+\left\{-s_{23}\tilde{\varepsilon}_{\mu\tau}^*\left(s_{13}e^{-i\delta}+\tilde{\varepsilon}_{e\tau}\right)+c_{23}\tilde{\varepsilon}_{\mu\tau}\left(c_{12}s_{12}\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{e\mu}\right)\right\}$$

$$\times r_A\left[\left(1-e^{-i\Delta L}\right)-\frac{1}{1-r_A}\left(e^{-ir_A\Delta L}-e^{-i\Delta L}\right)\right] \tag{A.3}$$

$$S_{\mu\mu} = c_{23}^2\left\{1-i(c_{12}^2 r_\Delta+\tilde{\varepsilon}_{\mu\mu}r_A)\Delta L\right\} + s_{23}^2\left(1-i\tilde{\varepsilon}_{\tau\tau}r_A\Delta L\right)e^{-i\Delta L}$$

$$-2c_{23}s_{23}\,\text{Re}(\tilde{\varepsilon}_{\mu\tau})r_A\left(1-e^{-i\Delta L}\right)$$

$$-s_{23}^2 s_{13}^2\left[(ir_A\Delta L)e^{-i\Delta L}-\frac{1+r_A}{1-r_A}\left(e^{-ir_A\Delta L}-e^{-i\Delta L}\right)\right]$$

$$+2s_{23}^2 s_{13} \text{Re}(\tilde{\varepsilon}_{e\tau}e^{i\delta})\left[(ir_A\Delta L)e^{-i\Delta L} + \frac{r_A}{1-r_A}\left(e^{-ir_A\Delta L} - e^{-i\Delta L}\right)\right]$$

$$+2c_{23}s_{23}s_{13}\ \text{Re}(\tilde{\varepsilon}_{e\mu}e^{i\delta})r_A\left(1 - e^{-i\Delta L}\right)$$

$$+2c_{23}s_{23}s_{13}\left\{c_{12}s_{12}\cos\delta\frac{r_\Delta}{r_A} + \ \text{Re}\left(\tilde{\varepsilon}_{e\mu}e^{i\delta}\right)\right\}\left(1 - e^{-ir_A\Delta L}\right)$$

$$-\left[c_{23}^2(c_{12}^2\frac{r_\Delta}{r_A} + \tilde{\varepsilon}_{\mu\mu})^2 + s_{23}^2\tilde{\varepsilon}_{\tau\tau}^2 e^{-i\Delta L}\right]\frac{(r_A\Delta L)^2}{2}$$

$$+c_{23}^2\left[|c_{12}s_{12}\frac{r_\Delta}{r_A} + \tilde{\varepsilon}_{e\mu}|^2\left\{(ir_A\Delta L) - \left(1 - e^{-ir_A\Delta L}\right)\right\} - |\tilde{\varepsilon}_{\mu\tau}|^2 r_A^2\left(1 - i\Delta L - e^{-i\Delta L}\right)\right]$$

$$-s_{23}^2\left[|s_{13}e^{-i\delta} + \tilde{\varepsilon}_{e\tau}|^2\left(\frac{r_A^2}{1-r_A}\right)\left\{i\Delta L e^{-i\Delta L} - \frac{1}{1-r_A}\left(e^{-ir_A\Delta L} - e^{-i\Delta L}\right)\right\}\right.$$

$$\left.+|\tilde{\varepsilon}_{\mu\tau}|^2 r_A^2\left\{i\Delta L e^{-i\Delta L} - (1 - e^{-i\Delta L})\right\}\right]$$

$$+2c_{23}s_{23}\text{Re}(\tilde{\varepsilon}_{\mu\tau})r_A^2\left[i\Delta L\left(c_{12}^2\frac{r_\Delta}{r_A} + \tilde{\varepsilon}_{\mu\mu} - \tilde{\varepsilon}_{\tau\tau}e^{-i\Delta L}\right)\right.$$

$$\left.-\left(c_{12}^2\frac{r_\Delta}{r_A} + \tilde{\varepsilon}_{\mu\mu} - \tilde{\varepsilon}_{\tau\tau}\right)\left(1 - e^{-i\Delta L}\right)\right]$$

$$+2c_{23}s_{23}\text{Re}\left\{\left(c_{12}s_{12}\frac{r_\Delta}{r_A} + \tilde{\varepsilon}_{e\mu}^*\right)\left(s_{13}e^{-i\delta} + \tilde{\varepsilon}_{e\tau}\right)\right\}$$

$$\times r_A\left[(1 - e^{-i\Delta L}) - \frac{1}{1-r_A}\left(e^{-ir_A\Delta L} - e^{-i\Delta L}\right)\right] \tag{A.4}$$

$$S_{\tau\tau} = s_{23}^2\left\{1 - i(c_{12}^2 r_\Delta + \tilde{\varepsilon}_{\mu\mu}r_A)\Delta L\right\} + c_{23}^2\left(1 - i\tilde{\varepsilon}_{\tau\tau}r_A\Delta L\right)e^{-i\Delta L}$$

$$+2c_{23}s_{23}\ \text{Re}(\tilde{\varepsilon}_{\mu\tau})r_A\left(1 - e^{-i\Delta L}\right)$$

$$-c_{23}^2 s_{13}^2\left[(ir_A\Delta L)e^{-i\Delta L} - \frac{1+r_A}{1-r_A}\left(e^{-ir_A\Delta L} - e^{-i\Delta L}\right)\right]$$

$$+2c_{23}^2 s_{13}\text{Re}(\tilde{\varepsilon}_{e\tau}e^{i\delta})\left[(ir_A\Delta L)e^{-i\Delta L} + \frac{r_A}{1-r_A}\left(e^{-ir_A\Delta L} - e^{-i\Delta L}\right)\right]$$

$$-2c_{23}s_{23}s_{13}\ \text{Re}(\tilde{\varepsilon}_{e\mu}e^{i\delta})r_A\left(1 - e^{-i\Delta L}\right)$$

$$-2c_{23}s_{23}s_{13}\left\{c_{12}s_{12}\cos\delta\frac{r_\Delta}{r_A} + \ \text{Re}\left(\tilde{\varepsilon}_{e\mu}e^{i\delta}\right)\right\}\left(1 - e^{-ir_A\Delta L}\right)$$

$$-\left[s_{23}^2(c_{12}^2\frac{r_\Delta}{r_A} + \tilde{\varepsilon}_{\mu\mu})^2 + c_{23}^2\tilde{\varepsilon}_{\tau\tau}^2 e^{-i\Delta L}\right]\frac{(r_A\Delta L)^2}{2}$$

$$+s_{23}^2\left[|c_{12}s_{12}\frac{r_\Delta}{r_A} + \tilde{\varepsilon}_{e\mu}|^2\left\{(ir_A\Delta L) - \left(1 - e^{-ir_A\Delta L}\right)\right\} - |\tilde{\varepsilon}_{\mu\tau}|^2 r_A^2\left(1 - e^{-i\Delta L}\right)\right]$$

$$-c_{23}^2\left[|s_{13}e^{-i\delta} + \tilde{\varepsilon}_{e\tau}|^2\left(\frac{r_A^2}{1-r_A}\right)\left\{i\Delta L e^{-i\Delta L} - \frac{1}{1-r_A}\left(e^{-ir_A\Delta L} - e^{-i\Delta L}\right)\right\}\right.$$

$$\left.+|\tilde{\varepsilon}_{\mu\tau}|^2 r_A^2\left\{i\Delta L e^{-i\Delta L} - (1 - i\Delta L - e^{-i\Delta L})\right\}\right]$$

$$-2c_{23}s_{23}\text{Re}(\tilde{\varepsilon}_{\mu\tau})r_A^2\left[i\Delta L\left(c_{12}^2\frac{r_\Delta}{r_A} + \tilde{\varepsilon}_{\mu\mu} - \tilde{\varepsilon}_{\tau\tau}e^{-i\Delta L}\right)\right.$$

$$\left.-\left(c_{12}^2\frac{r_\Delta}{r_A} + \tilde{\varepsilon}_{\mu\mu} - \tilde{\varepsilon}_{\tau\tau}\right)\left(1 - e^{-i\Delta L}\right)\right]$$

$$-2c_{23}s_{23}\text{Re}\left\{\left(c_{12}s_{12}\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{e\mu}^*\right)\left(s_{13}e^{-i\delta}+\tilde{\varepsilon}_{e\tau}\right)\right\}$$

$$\times r_A\left[(1-e^{-i\Delta L})-\frac{1}{1-r_A}\left(e^{-ir_A\Delta L}-e^{-i\Delta L}\right)\right] \tag{A.5}$$

$$\begin{aligned}
S_{\mu\tau} = &-c_{23}s_{23}\left\{1-i(c_{12}^2 r_\Delta+\tilde{\varepsilon}_{\mu\mu}r_A)\Delta L\right\}+c_{23}s_{23}\left(1-i\tilde{\varepsilon}_{\tau\tau}r_A\Delta L\right)e^{-i\Delta L}\\
&-\left\{(c_{23}^2-s_{23}^2)\text{Re}(\tilde{\varepsilon}_{\mu\tau})-i\,\text{Im}(\tilde{\varepsilon}_{\mu\tau})\right\}r_A\left(1-e^{-i\Delta L}\right)\\
&-c_{23}s_{23}s_{13}^2\left[(ir_A\Delta L)e^{-i\Delta L}-\frac{1+r_A}{1-r_A}\left(e^{-ir_A\Delta L}-e^{-i\Delta L}\right)\right]\\
&+2c_{23}s_{23}s_{13}\text{Re}(\tilde{\varepsilon}_{e\tau}e^{i\delta})\left[(ir_A\Delta L)e^{-i\Delta L}+\frac{r_A}{1-r_A}\left(e^{-ir_A\Delta L}-e^{-i\Delta L}\right)\right]\\
&+s_{13}\left\{(c_{23}^2-s_{23}^2)\text{Re}(\tilde{\varepsilon}_{e\mu}e^{i\delta})+i\,\text{Im}(\tilde{\varepsilon}_{e\mu}e^{i\delta})\right\}r_A\left(1-e^{-i\Delta L}\right)\\
&+s_{13}\left[(c_{23}^2-s_{23}^2)\text{Re}\left\{e^{i\delta}\left(c_{12}s_{12}\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{e\mu}\right)\right\}-i\,\text{Im}\left\{e^{i\delta}\left(c_{12}s_{12}\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{e\mu}\right)\right\}\right]\left(1-e^{-ir_A\Delta L}\right)\\
&+r_A^2\left(c_{23}^2\tilde{\varepsilon}_{\mu\tau}-s_{23}^2\tilde{\varepsilon}_{\mu\tau}^*\right)\left[i\Delta L\left(c_{12}^2\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{\mu\mu}-\tilde{\varepsilon}_{\tau\tau}e^{-i\Delta L}\right)\right.\\
&\qquad\qquad\left.-\left(c_{12}^2\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{\mu\mu}-\tilde{\varepsilon}_{\tau\tau}\right)\left(1-e^{-i\Delta L}\right)\right]\\
&+\left[c_{23}^2\left(c_{12}s_{12}\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{e\mu}^*\right)\left(s_{13}e^{-i\delta}+\tilde{\varepsilon}_{e\tau}\right)-s_{23}^2\left(c_{12}s_{12}\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{e\mu}\right)\left(s_{13}e^{i\delta}+\tilde{\varepsilon}_{e\tau}^*\right)\right]\\
&\times r_A\left[(1-e^{-i\Delta L})-\frac{1}{1-r_A}\left(e^{-ir_A\Delta L}-e^{-i\Delta L}\right)\right]\\
&+c_{23}s_{23}\left[\left(c_{12}^2\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{\mu\mu}\right)^2-\tilde{\varepsilon}_{\tau\tau}^2 e^{-i\Delta L}\right]\frac{(r_A\Delta L)^2}{2}\\
&-c_{23}s_{23}|s_{13}e^{-i\delta}+\tilde{\varepsilon}_{e\tau}|^2\left(\frac{r_A^2}{1-r_A}\right)\left[i\Delta L e^{-i\Delta L}-\frac{1}{1-r_A}\left(e^{-ir_A\Delta L}-e^{-i\Delta L}\right)\right]\\
&-c_{23}s_{23}|c_{12}s_{12}\frac{r_\Delta}{r_A}+\tilde{\varepsilon}_{e\mu}|^2\left\{ir_A\Delta L-\left(1-e^{-ir_A\Delta L}\right)\right\}\\
&-c_{23}s_{23}|\tilde{\varepsilon}_{\mu\tau}|^2 r_A^2\left\{i\Delta L(1+e^{-i\Delta L})-2\left(1-e^{-ir_A\Delta L}\right)\right\} \tag{A.6}
\end{aligned}$$

The other S matrix elements are given by either the T-conjugate relations

$$\begin{aligned}
S_{\mu e}(\delta,\phi_{\alpha\beta}) &= S_{e\mu}(-\delta,-\phi_{\alpha\beta}),\\
S_{\tau e}(\delta,\phi_{\alpha\beta}) &= S_{e\tau}(-\delta,-\phi_{\alpha\beta}),\\
S_{\tau\mu}(\delta,\phi_{\alpha\beta}) &= S_{\mu\tau}(-\delta,-\phi_{\alpha\beta}),
\end{aligned} \tag{A.7}$$

or by the CP-conjugate relations for antineutrino channels

$$\begin{aligned}
\bar{S}_{e\mu}(\delta,\phi_{\alpha\beta},a) &= S_{e\mu}(-\delta,-\phi_{\alpha\beta},-a),\\
\bar{S}_{e\tau}(\delta,\phi_{\alpha\beta},a) &= S_{e\tau}(-\delta,-\phi_{\alpha\beta},-a),\\
\bar{S}_{\mu\tau}(\delta,\phi_{\alpha\beta},a) &= S_{\mu\tau}(-\delta,-\phi_{\alpha\beta},-a).
\end{aligned} \tag{A.8}$$

B NSI second-order probability formulas

In this appendix we give the explicit expressions of the oscillation probabilities to second order in ϵ in all channels, except for those which can be readily obtained by the extended transformation (4.13).

B.1 Oscillation probability in the ν_e-related sector

We present here the explicit forms of $P(\nu_e \to \nu_e)$ and $P(\nu_e \to \nu_\mu)$ for completeness and possible convenience of the readers considering importance of the appearance channels.

$$
P(\nu_e \to \nu_e) = 1 - 4\left| c_{12}s_{12}\frac{\Delta m_{21}^2}{a} + c_{23}\varepsilon_{e\mu} - s_{23}\varepsilon_{e\tau} \right|^2 \sin^2 \frac{aL}{4E} \tag{B.1}
$$
$$
- 4\left| s_{13}e^{-i\delta}\frac{\Delta m_{31}^2}{a} + s_{23}\varepsilon_{e\mu} + c_{23}\varepsilon_{e\tau} \right|^2 \left(\frac{a}{\Delta m_{31}^2 - a} \right)^2 \sin^2 \frac{\Delta m_{31}^2 - a}{4E}L,
$$

$$
P(\nu_e \to \nu_\mu) = 4c_{23}^2 \left| c_{12}s_{12}\frac{\Delta m_{21}^2}{a} + c_{23}\varepsilon_{e\mu} - s_{23}\varepsilon_{e\tau} \right|^2 \sin^2 \frac{aL}{4E}
$$
$$
+ 4s_{23}^2 \left| s_{13}e^{-i\delta}\frac{\Delta m_{31}^2}{a} + s_{23}\varepsilon_{e\mu} + c_{23}\varepsilon_{e\tau} \right|^2 \left(\frac{a}{\Delta m_{31}^2 - a} \right)^2 \sin^2 \frac{\Delta m_{31}^2 - a}{4E}L
$$
$$
+ 8c_{23}s_{23}\ \text{Re}\left[(c_{12}s_{12}\frac{\Delta m_{21}^2}{a} + c_{23}\varepsilon_{e\mu} - s_{23}\varepsilon_{e\tau})(s_{13}e^{i\delta}\frac{\Delta m_{31}^2}{a} + s_{23}\varepsilon_{e\mu}^* + c_{23}\varepsilon_{e\tau}^*) \right]
$$
$$
\times \frac{a}{\Delta m_{31}^2 - a} \sin \frac{aL}{4E} \cos \frac{\Delta m_{31}^2 L}{4E} \sin \frac{\Delta m_{31}^2 - a}{4E}L
$$
$$
+ 8c_{23}s_{23}\ \text{Im}\left[(c_{12}s_{12}\frac{\Delta m_{21}^2}{a} + c_{23}\varepsilon_{e\mu} - s_{23}\varepsilon_{e\tau})(s_{13}e^{i\delta}\frac{\Delta m_{31}^2}{a} + s_{23}\varepsilon_{e\mu}^* + c_{23}\varepsilon_{e\tau}^*) \right]
$$
$$
\times \frac{a}{\Delta m_{31}^2 - a} \sin \frac{aL}{4E} \sin \frac{\Delta m_{31}^2 L}{4E} \sin \frac{\Delta m_{31}^2 - a}{4E}L. \tag{B.2}
$$

$P(\nu_e \to \nu_\tau)$ can be obtained from $P(\nu_e \to \nu_\mu)$ by the transformation (4.13). Or, the simpler way of remembering the operation is to do transformation $c_{23} \to -s_{23}$ and $s_{23} \to c_{23}$ in $P(\nu_e \to \nu_\mu)$, but undoing any transformation in the generalized atmospheric and the solar variables defined in (6.5), the pieces bracketed in the real and imaginary parts in (B.2). (See also (B.3)).

B.2 Oscillation probability in the $\nu_\mu - \nu_\tau$ sector

For compact expressions of the oscillation probabilities in the $\nu_\mu - \nu_\tau$ sector, we define the simplified notations which involve ε's in the $\nu_\mu - \nu_\tau$ sector as well as ε_{ee}.[20] Together with the ones already defined in section 6.1, they are as follows:

$$
\Theta_\pm \equiv s_{13}\frac{\Delta m_{31}^2}{a} + (s_{23}\varepsilon_{e\mu} + c_{23}\varepsilon_{e\tau})e^{i\delta} \equiv |\Theta_\pm|e^{i\theta_\pm},
$$
$$
\Xi \equiv \left(c_{12}s_{12}\frac{\Delta m_{21}^2}{a} + c_{23}\varepsilon_{e\mu} - s_{23}\varepsilon_{e\tau} \right) e^{i\delta} \equiv |\Xi|e^{i\xi},
$$

---

[20]For readers who want to see the fully explicit expressions of all the oscillation probabilities, we refer the first arXiv version of this paper [68].

$$\mathcal{E} \equiv c_{23}s_{23}(\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}) + c_{23}^2\varepsilon_{\mu\tau} - s_{23}^2\varepsilon_{\mu\tau}^* \equiv |\mathcal{E}|e^{i\phi},$$

$$\mathcal{S}_1 \equiv (c_{23}^2 - s_{23}^2)(\varepsilon_{\tau\tau} - \varepsilon_{\mu\mu}) + 2c_{23}s_{23}(\varepsilon_{\mu\tau} + \varepsilon_{\mu\tau}^*) - c_{12}^2\frac{\Delta m_{21}^2}{a},$$

$$\mathcal{S}_2 \equiv (\varepsilon_{\mu\mu} - \varepsilon_{ee}) + c_{23}^2(\varepsilon_{\tau\tau} - \varepsilon_{\mu\mu}) + c_{23}s_{23}(\varepsilon_{\mu\tau} + \varepsilon_{\mu\tau}^*) - s_{12}^2\frac{\Delta m_{21}^2}{a},$$

$$\mathcal{S}_3 \equiv (\varepsilon_{\mu\mu} - \varepsilon_{ee}) + s_{23}^2(\varepsilon_{\tau\tau} - \varepsilon_{\mu\mu}) - c_{23}s_{23}(\varepsilon_{\mu\tau} + \varepsilon_{\mu\tau}^*) + (c_{12}^2 - s_{12}^2)\frac{\Delta m_{21}^2}{a}. \quad \text{(B.3)}$$

Notice that $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ are not independent, $\mathcal{S}_1 = \mathcal{S}_2 - \mathcal{S}_3$. We also note that $\Theta_\pm$, Ξ, and $\mathcal{E}$ are complex numbers while the others are real.

To present the oscillation probabilities in the $\nu_\mu - \nu_\tau$ sector, we start by recapitulating the decomposition formula (6.18) in section 6.2:

$$P(\nu_\alpha \to \nu_\beta; \varepsilon_{e\mu}, \varepsilon_{e\tau}, \varepsilon_{\mu\mu}, \varepsilon_{\mu\tau}, \varepsilon_{\tau\tau}) = P(\nu_\alpha \to \nu_\beta; 2 \text{ flavor in vacuum})$$
$$+ P(\nu_\alpha \to \nu_\beta; \varepsilon_{e\mu}, \varepsilon_{e\tau})$$
$$+ P(\nu_\alpha \to \nu_\beta; \varepsilon_{\mu\mu}, \varepsilon_{\mu\tau}, \varepsilon_{\tau\tau}) \quad \text{(B.4)}$$

where α and β denote one of μ and τ. The first term in (B.4) has a form that it appears in the two flavor oscillation in vacuum:

$$P(\nu_\mu \to \nu_\mu; 2 \text{ flavor in vacuum}) = P(\nu_\tau \to \nu_\tau; 2 \text{ flavor in vacuum})$$
$$= 1 - 4c_{23}^2 s_{23}^2 \sin^2\frac{\Delta m_{31}^2 L}{4E},$$
$$P(\nu_\mu \to \nu_\tau; 2 \text{ flavor in vacuum}) = 4c_{23}^2 s_{23}^2 \sin^2\frac{\Delta m_{31}^2 L}{4E}. \quad \text{(B.5)}$$

We have shown in section 6.2 that the third term in (B.4) in the $\nu_\mu \to \nu_\mu$, $\nu_\tau \to \nu_\tau$, and $\nu_\mu \to \nu_\tau$ channels are given by the single equation[21]

$$P(\nu_\mu \to \nu_\mu; \varepsilon_{\mu\mu}, \varepsilon_{\mu\tau}, \varepsilon_{\tau\tau})$$
$$= P(\nu_\tau \to \nu_\tau; \varepsilon_{\mu\mu}, \varepsilon_{\mu\tau}, \varepsilon_{\tau\tau}) = -P(\nu_\mu \to \nu_\tau; \varepsilon_{\mu\mu}, \varepsilon_{\mu\tau}, \varepsilon_{\tau\tau})$$
$$= 2c_{23}^2 s_{23}^2 \left(s_{13}^2\frac{\Delta m_{31}^2}{a} - \mathcal{S}_1\right)\left(\frac{aL}{2E}\right)\sin\frac{\Delta m_{31}^2 L}{2E} - c_{23}^2 s_{23}^2 \mathcal{S}_1^2\left(\frac{aL}{2E}\right)^2\cos\frac{\Delta m_{31}^2 L}{2E}$$
$$+ 8c_{23}s_{23}(c_{23}^2 - s_{23}^2)\left[c_{12}s_{12}s_{13}\cos\delta\left(\frac{\Delta m_{21}^2}{a}\right) - |\mathcal{E}|\cos\phi\right]\left(\frac{a}{\Delta m_{31}^2}\right)\sin^2\frac{\Delta m_{31}^2 L}{4E}$$
$$- 4c_{23}s_{23}(c_{23}^2 - s_{23}^2)\mathcal{S}_1|\mathcal{E}|\cos\phi\left(\frac{a}{\Delta m_{31}^2}\right)\left[\left(\frac{aL}{2E}\right)\sin\frac{\Delta m_{31}^2 L}{2E} - 2\left(\frac{a}{\Delta m_{31}^2}\right)\sin^2\frac{\Delta m_{31}^2 L}{4E}\right]$$
$$- 4c_{23}^2 s_{23}^2|\mathcal{E}|^2\left(\frac{a}{\Delta m_{31}^2}\frac{aL}{2E}\right)\sin\frac{\Delta m_{31}^2 L}{2E}$$
$$- 4|\mathcal{E}|^2\left[(c_{23}^2 - s_{23}^2)^2 - 4c_{23}^2 s_{23}^2\cos^2\phi\right]\left(\frac{a}{\Delta m_{31}^2}\right)^2\sin^2\frac{\Delta m_{31}^2 L}{4E}. \quad \text{(B.6)}$$

---

[21] To second order in ϵ the sensitivity to $\varepsilon_{\mu\mu}$ and $\varepsilon_{\tau\tau}$ is through the form $\varepsilon_{\mu\mu} - \varepsilon_{\tau\tau}$, and hence no sensitivity to the individual ε's. Generally, the diagonal ε's appear in a form of difference in the oscillation probabilities as one can observe in the third-order formula given in appendix C. It must be the case because the over-all phase is an unobservable.

The second term in (B.4) is given in $\nu_\mu \to \nu_\mu$ channel as

$$P(\nu_\mu \to \nu_\mu; \varepsilon_{e\mu}, \varepsilon_{e\tau}) = -4c_{23}^2 |\Xi|^2 \sin^2 \frac{aL}{4E} - 2c_{23}^2 s_{23}^2 |\Xi|^2 \left(\frac{aL}{2E}\right) \sin \frac{\Delta m_{31}^2 L}{2E} \tag{B.7}$$

$$+ 8c_{23}^2 s_{23}^2 |\Xi|^2 \sin \frac{aL}{4E} \sin \frac{\Delta m_{31}^2 L}{4E} \cos \frac{\Delta m_{31}^2 - a}{4E} L$$

$$- 4s_{23}^2 |\Theta_\pm|^2 \left(\frac{a}{\Delta m_{31}^2 - a}\right)^2 \sin^2 \frac{\Delta m_{31}^2 - a}{4E} L - 2c_{23}^2 s_{23}^2 |\Theta_\pm|^2 \left(\frac{a}{\Delta m_{31}^2 - a} \frac{aL}{2E}\right) \sin \frac{\Delta m_{31}^2 L}{2E}$$

$$+ 8c_{23}^2 s_{23}^2 |\Theta_\pm|^2 \left(\frac{a}{\Delta m_{31}^2 - a}\right)^2 \cos \frac{aL}{4E} \sin \frac{\Delta m_{31}^2 L}{4E} \sin \frac{\Delta m_{31}^2 - a}{4E} L$$

$$+ 8c_{23} s_{23} |\Xi| |\Theta_\pm| \cos(\xi - \theta_\pm) \left(\frac{a}{\Delta m_{31}^2 - a}\right)$$

$$\times \left[c_{23}^2 \sin^2 \frac{aL}{4E} + s_{23}^2 \sin^2 \frac{\Delta m_{31}^2 - a}{4E} L - s_{23}^2 \sin^2 \frac{\Delta m_{31}^2 L}{4E} - (c_{23}^2 - s_{23}^2) \frac{a}{\Delta m_{31}^2} \sin^2 \frac{\Delta m_{31}^2 L}{4E} \right].$$

The subscript $\pm$ in this and the following equations denotes the normal and the inverted mass hierarchies, which corresponds to the positive and negative values of Δm_{31}^2. Notice again that $P(\nu_\tau \to \nu_\tau; \varepsilon_{e\mu}, \varepsilon_{e\tau})$ can be obtained from $P(\nu_\mu \to \nu_\mu; \varepsilon_{e\mu}, \varepsilon_{e\tau})$ by the extended transformation (4.13), or by the operation described at the end of the previous subsection.

Finally, the second term in the oscillation probability in the $\nu_\mu \to \nu_\tau$ channel is given by

$$P(\nu_\mu \to \nu_\tau; \varepsilon_{e\mu}, \varepsilon_{e\tau})$$

$$= 4c_{23}^2 s_{23}^2 |\Xi|^2 \left(\frac{aL}{4E}\right) \sin \frac{\Delta m_{31}^2 L}{2E} - 8c_{23}^2 s_{23}^2 |\Xi|^2 \sin \frac{aL}{4E} \sin \frac{\Delta m_{31}^2 L}{4E} \cos \frac{\Delta m_{31}^2 - a}{4E} L$$

$$+ 4c_{23}^2 s_{23}^2 |\Theta_\pm|^2 \left(\frac{a}{\Delta m_{31}^2 - a}\right) \left(\frac{aL}{4E}\right) \sin \frac{\Delta m_{31}^2 L}{2E}$$

$$- 8c_{23}^2 s_{23}^2 |\Theta_\pm|^2 \left(\frac{a}{\Delta m_{31}^2 - a}\right)^2 \cos \frac{aL}{4E} \sin \frac{\Delta m_{31}^2 L}{4E} \sin \frac{\Delta m_{31}^2 - a}{4E} L$$

$$+ 8c_{23} s_{23} (c_{23}^2 - s_{23}^2) |\Xi| |\Theta_\pm| \cos(\xi - \theta_\pm) \left(\frac{a}{\Delta m_{31}^2 - a}\right) \left(\frac{a}{\Delta m_{31}^2}\right) \sin^2 \frac{\Delta m_{31}^2 L}{4E}$$

$$+ 8c_{23} s_{23} |\Xi| |\Theta_\pm| \left(\frac{a}{\Delta m_{31}^2 - a}\right) \sin \frac{aL}{4E} \sin \frac{\Delta m_{31}^2 L}{4E}$$

$$\times \left[s_{23}^2 \cos\left(\xi - \theta_\pm - \frac{\Delta m_{31}^2 - a}{4E} L\right) - c_{23}^2 \cos\left(\xi - \theta_\pm + \frac{\Delta m_{31}^2 - a}{4E} L\right) \right]. \tag{B.8}$$

C NSI third-order formula

We present here the third-order formula for the oscillation probability with NSI. Though utility of such lengthy formula may be subject to doubt we can offer at least three arguments to justify the presentation of the formula in this appendix. Firstly, the third-order formula is needed to complete table 1. Secondly, by turning off all the NSI elements one can obtain the SI third-order formula for the oscillation probabilities with SI only, which is valid to

order ϵ^3. To our knowledge, such formula has never been derived in the literature. Utility of the SI third-order formula for theoretical analysis may be obvious if the sensitivity to the oscillation probability reaches to the level of $\sim 10^{-5}$, which is smaller than terms of order ϵ^2. In fact, it appears to be the case in some of the future facilities according to the analysis in [69]. Thirdly, once the sensitivity to the oscillation probability comes down to $\sim 10^{-5}$, a complete treatment of neutrino oscillation probability must include NSI elements up to the same order as θ_{13} and $\Delta m_{21}^2/\Delta m_{31}^2$, as far as our ansatz (5.1) in formulating the ϵ perturbation theory is correct. Thus, we believe that the NSI third-order formula has a good chance to be useful.

In presenting the third-order probability formula we restrict ourselves to the ν_e related channel, and only present $P(\nu_e \to \nu_\mu)$ here because from which $P(\nu_e \to \nu_\tau)$ can be obtained by the extended transformation (4.13). Then, $P(\nu_e \to \nu_e)$ can be readily calculated by using the unitarity relation. The NSI third-order formula for $P(\nu_e \to \nu_\mu)$ reads

$$
\begin{aligned}
P(\nu_e \to \nu_\mu) =\ & 4c_{23}^2|\Xi|^2 \sin^2 \frac{aL}{4E} + 4s_{23}^2|\Theta_\pm|^2\left(\frac{a}{\Delta m_{31}^2-a}\right)^2 \sin^2 \frac{\Delta m_{31}^2-a}{4E}L \qquad\qquad \text{(C.1)}\\
& +8c_{23}s_{23}|\Xi||\Theta_\pm| \cos\left(\xi-\theta_\pm - \frac{\Delta m_{31}^2 L}{4E}\right)\left(\frac{a}{\Delta m_{31}^2-a}\right)\sin\frac{aL}{4E}\sin\frac{\Delta m_{31}^2-a}{4E}L\\
& -8c_{23}s_{23}s_{12}^2 s_{13}|\Xi|\cos\left(\xi - \frac{\Delta m_{31}^2 L}{4E}\right)\left(\frac{\Delta m_{21}^2}{a}\right)\left(\frac{a}{\Delta m_{31}^2-a}\right)\sin\frac{aL}{4E}\sin\frac{\Delta m_{31}^2-a}{4E}L\\
& -8s_{23}^2 s_{12}^2 s_{13}|\Theta_\pm|\cos\theta_\pm\left(\frac{\Delta m_{21}^2}{a}\right)\left(\frac{a}{\Delta m_{31}^2-a}\right)^2\sin^2\frac{\Delta m_{31}^2-a}{4E}L\\
& +4s_{23}^2|\Theta_\pm|^2 \mathcal{S}_2 \left(\frac{a}{\Delta m_{31}^2-a}\right)^2\left[\left(\frac{aL}{4E}\right)\sin\frac{\Delta m_{31}^2-a}{2E}L - 2\left(\frac{a}{\Delta m_{31}^2-a}\right)\sin^2\frac{\Delta m_{31}^2-a}{4E}L\right]\\
& +4c_{23}^2|\Xi|^2 \mathcal{S}_3\left[2\sin^2\frac{aL}{4E}-\left(\frac{aL}{4E}\right)\sin\frac{aL}{2E}\right]\\
& +8c_{23}s_{23}|\Xi|^2|\mathcal{E}|\cos\left(\phi+\frac{\Delta m_{31}^2-a}{4E}L\right)\left(\frac{a^2}{\Delta m_{31}^2(\Delta m_{31}^2-a)}\right)\sin\frac{aL}{4E}\sin\frac{\Delta m_{31}^2 L}{4E}\\
& -8c_{23}s_{23}|\Xi|^2|\mathcal{E}|\cos\phi\left(\frac{a}{\Delta m_{31}^2-a}\right)\sin^2\frac{aL}{4E}\\
& +8c_{23}s_{23}|\Theta_\pm|^2|\mathcal{E}|\cos\phi\left(\frac{a}{\Delta m_{31}^2-a}\right)^2\sin^2\frac{\Delta m_{31}^2-a}{4E}L\\
& -8c_{23}s_{23}|\Theta_\pm|^2|\mathcal{E}|\cos\left(\phi+\frac{aL}{4E}\right)\left(\frac{a^2}{\Delta m_{31}^2(\Delta m_{31}^2-a)}\right)\sin\frac{\Delta m_{31}^2 L}{4E}\sin\frac{\Delta m_{31}^2-a}{4E}L\\
& -4c_{23}s_{23}|\Xi||\Theta_\pm|\left(\frac{a}{\Delta m_{31}^2-a}\right)\left(\frac{aL}{4E}\right)\\
& \times\left[\mathcal{S}_1\sin\left(\xi-\theta_\pm - \frac{\Delta m_{31}^2 L}{2E}\right)+\mathcal{S}_3\sin\left(\xi-\theta_\pm-\frac{aL}{2E}\right)-\mathcal{S}_2\sin\left(\xi-\theta_\pm-\frac{\Delta m_{31}^2-a}{2E}L\right)\right]\\
& -8c_{23}s_{23}|\Xi||\Theta_\pm|\cos\left(\xi-\theta_\pm-\frac{\Delta m_{31}^2 L}{4E}\right)\\
& \times\left[\left(\frac{a}{\Delta m_{31}^2-a}\right)\mathcal{S}_2 - \mathcal{S}_3\right]\left(\frac{a}{\Delta m_{31}^2-a}\right)\sin\frac{aL}{4E}\sin\frac{\Delta m_{31}^2-a}{4E}L\\
& -8s_{23}^2|\Xi||\Theta_\pm||\mathcal{E}|\cos\left(\xi+\phi-\theta_\pm-\frac{aL}{4E}\right)\left(\frac{a^2}{\Delta m_{31}^2(\Delta m_{31}^2-a)}\right)\sin\frac{\Delta m_{31}^2 L}{4E}\sin\frac{\Delta m_{31}^2-a}{4E}L
\end{aligned}
$$

$$+8c_{23}^2|\Xi||\Theta_\pm||\mathcal{E}|\cos\left(\xi+\phi-\theta_\pm-\frac{\Delta m_{31}^2-a}{4E}L\right)\left(\frac{a^2}{\Delta m_{31}^2(\Delta m_{31}^2-a)}\right)\sin\frac{aL}{4E}\sin\frac{\Delta m_{31}^2 L}{4E}$$

$$+8|\Xi||\Theta_\pm||\mathcal{E}|\cos\left(\xi+\phi-\theta_\pm\right)\left(\frac{a}{\Delta m_{31}^2-a}\right)\left[s_{23}^2\left(\frac{a}{\Delta m_{31}^2-a}\right)\sin^2\frac{\Delta m_{31}^2-a}{4E}L-c_{23}^2\sin^2\frac{aL}{4E}\right].$$

D Intrinsic degeneracy in vacuum

We re-examine the problem of intrinsic degeneracy in vacuum. For simplicity, we focus on the channel $\nu_\mu \to \nu_e$. We use a simplified notation $s_{13} \equiv s$ below. The neutrino and anti-neutrino oscillation probabilities in vacuum are given by

$$P(\nu_\mu \to \nu_e) = Xs^2 + (Y_c \cos\delta - Y_s \sin\delta)\,s + P_\odot$$
$$P(\bar\nu_\mu \to \bar\nu_e) = Xs^2 + (Y_c \cos\delta + Y_s \sin\delta)\,s + P_\odot \tag{D.1}$$

where X, Y's, etc. are defined with simplified symbol $\Delta_{ji} \equiv \frac{\Delta m_{ji}^2 L}{4E}$ as

$$X \equiv 4s_{23}^2 \sin^2 \Delta_{31},$$
$$Y_c \equiv \sin 2\theta_{12} \sin 2\theta_{23} \Delta_{21} \sin 2\Delta_{31},$$
$$Y_s \equiv 2\sin 2\theta_{12}\sin 2\theta_{23}\Delta_{21}\sin^2\Delta_{31},$$
$$P_\odot \equiv \sin^2 2\theta_{12} c_{23}^2 \Delta_{21}^2. \tag{D.2}$$

Let us denote two set of intrinsic degenerate solutions as (s_1, δ_1) and (s_2, δ_2). They satisfy

$$P - P_\odot = Xs_1^2 + (Y_c \cos\delta_1 - Y_s \sin\delta_1)\,s_1$$
$$P - P_\odot = Xs_2^2 + (Y_c \cos\delta_2 - Y_s \sin\delta_2)\,s_2 \tag{D.3}$$

and

$$\bar P - P_\odot = Xs_1^2 + (Y_c \cos\delta_1 + Y_s \sin\delta_1)\,s_1$$
$$\bar P - P_\odot = Xs_2^2 + (Y_c \cos\delta_2 + Y_s \sin\delta_2)\,s_2 \tag{D.4}$$

By subtracting two equations in (D.3) and (D.4) respectively, we obtain

$$X(s_1^2 - s_2^2) + Y_c(s_1 \cos\delta_1 - s_2 \cos\delta_2) - Y_s(s_1 \sin\delta_1 - s_2 \sin\delta_2) = 0,$$
$$X(s_1^2 - s_2^2) + Y_c(s_1 \cos\delta_1 - s_2 \cos\delta_2) + Y_s(s_1 \sin\delta_1 - s_2 \sin\delta_2) = 0. \tag{D.5}$$

They further simplifies to

$$s_1 \sin\delta_1 - s_2 \sin\delta_2 = 0, \tag{D.6}$$
$$X(s_1^2 - s_2^2) + Y_c(s_1 \cos\delta_1 - s_2 \cos\delta_2) = 0. \tag{D.7}$$

Equation (D.6) can be solved as

$$s_2 \cos\delta_2 = \pm\sqrt{s_2^2 - s_1^2 \sin^2\delta_1} \tag{D.8}$$

which can be inserted to (D.7) to yield the (formally quartic but actually) quadratic equation for s_2. Now, the issue here is to choose the correct sign in (D.8). One can show that by using (D.7) if $Y_c > 0$ ($Y_c < 0$), minus (plus) sign has to be chosen.

These equations can be easily solved for (s_2, δ_2) for given values of (s_1, δ_1) as inputs:

$$s_2 = \sqrt{s_1^2 + 2\left(\frac{Y_c}{X}\right)s_1\cos\delta_1 + \left(\frac{Y_c}{X}\right)^2}$$
$$\sin\delta_2 = \frac{s_1}{s_2}\sin\delta_1$$
$$\cos\delta_2 = \mp\frac{1}{s_2}\left(s_1\cos\delta_1 + \frac{Y_c}{X}\right) \tag{D.9}$$

where the sign $\mp$ for $\cos\delta_2$ are for $Y_c = \pm|Y_c|$, and s_2 in the solution of δ is meant to be the s_2 solution given in the first line in (D.9). By using

$$\frac{Y_c}{X} = \sin 2\theta_{12}\cot\theta_{23}\Delta_{21}\cot\Delta_{31} \tag{D.10}$$

s_2 can be written as

$$s_2 = \sqrt{s_1^2 + 2\sin 2\theta_{12}\cot\theta_{23}\Delta_{21}\cot\Delta_{31}s_1\cos\delta_1 + (\sin 2\theta_{12}\cot\theta_{23}\Delta_{21}\cot\Delta_{31})^2} \tag{D.11}$$

Similarly, $\cos\delta$ is given as

$$\cos\delta_2 = \mp\frac{1}{s_2}\left(s_1\cos\delta_1 + \sin 2\theta_{12}\cot\theta_{23}\Delta_{21}\cot\Delta_{31}\right) \tag{D.12}$$

By further expanding (D.9) by $\frac{Y_c}{X}$, assuming it small, the Burguet-Castell *et al.* solution [33, 57] is reproduced;

$$s_2 \simeq s_1 + \frac{Y_c}{X}\cos\delta_1 \tag{D.13}$$

Acknowledgments

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